Path-by-path regularization by noise for stochastic scalar conservation laws

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joint work with: Panagiotis E. Souganidis, Khalil Chouk
[G., Souganidis; CPAM, 2016], [Chouk, G.; arXiv]

Consider

$$\partial_t u + \mathrm{div} A(u) = 0, \quad \text{on } (0,T) \times \mathbb{R}^d \qquad (\star)$$
$$u(0) = u_0 \in L^\infty(\mathbb{R}^d).$$

For

$$\chi(t,x,v) = \chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}$$

we get the kinetic form

$$\partial_t \chi + A'(v) \cdot \nabla \chi = \partial_v m \quad \text{on } (0,T) \times \mathbb{R}^d \times \mathbb{R}.$$

Dissipation-dispersion approximations lead to

Definition (De Lellis, Otto, Westdickenberg, 2003)

A function $u \in L^\infty([0,T] \times \mathbb{R}^d)$ is said to be a quasi-solution if $\chi(t,x,v) = \chi(u(t,x),v)$ satisfies

$$\partial_t \chi + A'(v) \cdot \nabla \chi = \partial_v m \quad \text{on } (0,T) \times \mathbb{R}^d \times \mathbb{R}$$

for some finite (signed) measure m.

Theorem (De Lellis, Westdickenberg, 2003; Jabin, Perthame 2002)

Consider

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0, \quad \text{on } (0,T)\times\mathbb{R}.$$

Then

1. *Each quasi-solution satisfies, for all* $\lambda \in (0, \frac{1}{3})$,
$$u \in L^1([0,T]; W^{\lambda,1}(\mathbb{R})).$$
2. *For each* $\lambda > \frac{1}{3}$ *there exists a quasi-solution* u*, such that* u *is a weak solution and*
$$u \notin L^1([0,T]; W^{\lambda,1}(\mathbb{R})).$$

Regularity of solutions to stochastic SCL

- Consider mean field equations

$$dX_t^i = \sigma^L\left(X_t^i, \frac{1}{L}\sum_{j=1}^{L}\delta_{X_t^j}\right)\circ d\beta_t \quad \text{in } \mathbb{R}^N$$

Taking $L \to \infty$ and $\sigma^L \to \sigma$ leads to stochastic scalar conservation laws

$$d\pi + \operatorname{div}(\underbrace{\sigma(x,\pi)\pi}_{=:A(x,\pi)} \circ d\beta) = 0 \quad \text{on } (0,T)\times\mathbb{R}^d.$$

- Consider the case of A spatially homogeneous and truly nonlinear: I.e. there exist $\theta \in (0,1]$ and $C > 0$ such that, for all $\sigma \in S^{d-1}$, $z \in \mathbb{R}^d$ and $\varepsilon > 0$,

$$|\{v \in \mathbb{R} : |A'(v)\sigma - z| \le \varepsilon\}| \le C\varepsilon^\theta.$$

e.g. $A(u) = \frac{u^{l+1}}{l+1}$, then $\theta = \frac{1}{l}$, $l \ge 1$.

- For simplicity in this talk restrict to

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0.$$

Theorem (G., Souganidis; CPAM, 2016)

Let $u \in L^\infty$ be a quasi-solution to

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T}.$$

Then,

$$u \in L^1_t W^{\lambda,1}_x \quad \text{for all } \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.}.$$

If u is an entropy solution, then

$$u(t) \in W^{\lambda,1}_x \quad \text{for all } t > 0, \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.}. \qquad (\star)$$

Two resulting questions:

1. Can the zero set in $(\star)$ be chosen uniformly in t?
2. Characterize the properties of Brownian paths leading to $(\star)$.

Regularization by nonlinear noise

- Consider, for $w \in C([0,T])$,

$$du + \frac{1}{2}\partial_x u^2 \circ dw_t = 0, \quad \text{on } \mathbb{R}.$$

- Get

$$\|u(t)\|_{W_x^{1,\infty}} \le \Big(\max_{0\le s\le t}(w(s)-w(t)) \wedge (w(t) - \min_{0\le s\le t} w(s))\Big)^{-1}.$$

- Decisive path property: "Changing sign of the derivative".
- For $w = \beta$ we get

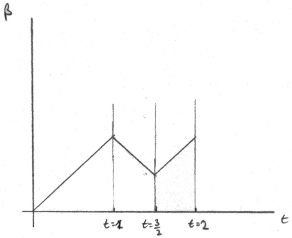
$$v(t) \in W^{1,\infty}, \quad \mathbb{P}-a.s.$$

- But: Zero set depends on time $t > 0$.

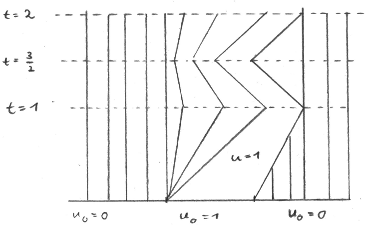
Regularization by nonlinear noise

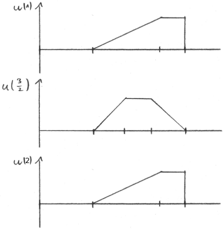
- Example:

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta = 0$$
$$u(0) = 1_{[0,1]}$$



- Solution u:





Path-by-path regularization by noise

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Framework

- Model example:
$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{T},$$
with $w \in C([0,T];\mathbb{R})$.
- Proof works in general dimension and general non-linear flux A.
- How to classify irregularity properties of w?

Idea of the proof

- Ideas of the proof of regularity for

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T}.$$

- By definition quasi-solutions satisfy

$$d\chi + v\partial_x \chi \circ d\beta_t = \partial_v m,$$

for some finite Radon measure m.

- Change of variables gives

$$\chi(t,x,v) = \chi_0(x + v\beta_t, v) + \int_0^t \partial_v m(s, x + v(\beta_t - \beta_s), v)ds.$$

- Averaging over velocity

$$u(t,x) = \int_v \chi = \int_v \chi_0(x + v\beta_t, v)dv + \int_0^t \int_v \partial_v m(s, x + v(\beta_t - \beta_s), v)dvds.$$

Framework

- Fourier transform in spatial variable

$$\hat{u}(t,n) = \int_v e^{-iv\beta_t n}\hat{\chi}_0(n,v)dv + \int_0^t \int_v e^{-iv(\beta_t-\beta_s)n}\partial_v \hat{m}(s,n,v)dvds.$$

- The oscillatory integrals have a regularizing effect, both in v and in $\beta_t - \beta_s$.
- For SDE this has been considered by [Catellier, Gubinelli; *SPA*, 2016]: A path $w \in C(\mathbb{R}_+;\mathbb{R}^d)$ is said to be (ρ,γ)-irregular if

$$|\int_s^t e^{i\langle a,w_r\rangle}dr| \lesssim (1+|a|)^{-\rho}|t-s|^\gamma \quad \forall a \in \mathbb{R}^d, s<t.$$

- Note:

$$\int_s^t e^{i\langle a,w_r\rangle}dr = \int_{\mathbb{R}} e^{i\langle a,x\rangle}dL_w^{s,t}(x) = \hat{L_w^{s,t}}(a)$$

the Fourier transform of the local time.

Main result

Theorem

Let $w \in C^{\eta}([0,T],\mathbb{R}^d)$ for some $\eta > 0$ be (ρ,γ)-irregular, u a bounded quasi-solution solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{T}.$$

Then, for all

$$\lambda < \frac{\rho(\eta+1)-(1-\gamma)}{(\rho \vee 1)(\eta+1)+(1-\gamma)},$$

we have

$$\|u\|_{L^1_t W^{\lambda,1}_x} < \infty.$$

Corollary

Let β^H be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}]$ and u be a bounded quasi-solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta_t^H = 0 \quad \text{on } \mathbb{T}. \tag{1}$$

Then, for all $\lambda < \frac{1}{1+2H}$,

$$\|u\|_{L_t^1 W_x^{\lambda,1}} < \infty.$$

- Note: Fully recover the probabilistic result from [G., Souganidis; *CPAM*, 2016]: For $H = \frac{1}{2}$ get $\lambda < \frac{1}{2}$.

A path-by-path scaling condition

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Discussion of the path classification

- The proof given in [G., Souganidis; *CPAM*, 2016] uses the *scaling property* of Brownian motion and independence of increments.
- However: (ρ, γ)-irregularity depends on two parameters, also encoding a time regularity. Hence, does not seem to be optimal.
- Moreover: (ρ, γ)-irregularity not easy to check.
- To avoid the use of oscillatory integrals: Completely avoid Fourier methods in the proof (motivated by [Jabin, Vega, *JMPA*, 2004]).

Path-by-path scaling condition

Leads to

- *Path-by-path scaling condition*: Assume that there is a $\iota \in [\frac{1}{2}, 1]$ such that for every $\sigma \in [0,1)$, $\lambda \geq 1$ we have

$$\int_0^T \int_0^{T-r} e^{-\lambda t} |\underbrace{w_{t+r} - w_r}_{=:w_{r,r+t}}|^{-\sigma} \, dtdr \lesssim \lambda^{-1+\iota\sigma}.$$

- Easy to see: (ρ, γ)-irregularity implies path-by-path scaling.
- Assuming path-by-path scaling can obtain the same regularity result as above.

Thanks

Thanks!