Finite speed of propagation for stochastic porous media equations.

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Outline

1. The deterministic case
2. The stochastic case
3. Random attractors

The deterministic case

The deterministic case

The Deterministic PME

- Recall: The deterministic porous medium equation

$$\frac{d}{dt}u = \Delta(u^m), \quad m > 1 \tag{1}$$

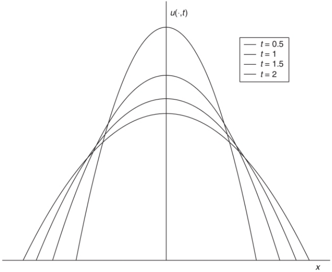
  for non-negative initial conditions $u_0 \geq 0$ [Vázquez, 2006]. For simplicity write u^m for $|u|^{m-1}u$.
- In the superlinear case $m > 1$, (1) has degenerate diffusivity:

$$\frac{d}{dt}u = mu^{m-1}\Delta u + m(m-1)u^{m-2}|\nabla u|^2.$$

  The diffusivity coefficient vanishes for $u \to 0$.

Finite speed of propagation

- Limited regularity: ∇u discontinuous. E.g. Barenblatt solutions:



- Regularity is limited precisely at the free boundary.
- Comparison to Barenblatt solutions yields finite speed of propagation.

Optimal estimates

- Finite speed of propagation is a local property.
- *Finite speed of hole-filling*:
  Let u_0 vanish in $B_R(x_0)$. Then u vanishes in $B_{R(t)}(x_0)$ with
  $$R(t) \geq R - \left(\frac{\|u\|_\infty^{m-1}}{C_{det}} t \right)^{1/2}.$$
  This is optimal.
- *Finite speed of propagation:*
  Let $S(t) := \{x \in \mathbb{R}^d | u(t,x) > 0\}$ be the positivity set. Then
  $$S(t+h) \subseteq B_{ch^{1/2}}(S(t)).$$

The stochastic case

The stochastic case

Stochastic porous medium equation

- Stochastic porous medium equation

$$dX_t = \Delta X_t^m dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}. \tag{SPME}$$

  on bounded domains, homogeneous Dirichlet boundary conditions.
- $z^{(k)}$ is a continuous semimartingale, $f_k \in C^\infty(\mathscr{O})$.

Known results

- *Finite speed of hole-filling* [Barbu, Röckner, EJP 2012]:
  Let X_0 vanish in $B_R(x_0)$. Then X vanishes in $B_{R(t,\omega)}(x_0)$ for some function $R(\cdot,\omega):[0,T]\to(0,R)$.
- No uniform control on $R(t,\omega)$ in x_0 $\to$ cannot deduce finite speed of propagation
- No information about optimality of the bounds
- Two aims:
  - show finite speed of propagation
  - deduce (locally) optimal bounds

Transformation

- Recall:

$$dX_t = \Delta(X_t^m)\,dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}.$$

- Set $Y_t := e^{\mu_t} X_t$, where $\mu_t = -\sum_{k=1}^{N} f_k \beta_t^{(k)}$. Then

$$\partial_t Y(t,x) = e^{\mu(t,x)} \Delta \left(e^{-\mu(t,x)} Y(t,x) \right)^m .$$

- Existence and uniqueness in [Barbu, Röckner, JDE, 2011], [G., to appear in AoP].

Non-spatially distributed noise

- Assume f_k constant. Then

$$\partial_t Y_t = e^{(1-m)\mu_t} \Delta Y_t^m.$$

- Let $F' := e^{(1-m)\mu_t}$, $g = F^{-1}$. Then $u_t := Y_{g(t)}$ solves

$$\frac{d}{dt} u = \Delta(u^m)$$

- Finite speed of propagation follows from the deterministic case and the estimates are **optimal**.
- *Finite speed of hole-filling:* Let X_0 vanish in $B_R(x_0)$. Then X_t vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$\begin{aligned} R_{stoch}(t) &= R - \left(\frac{H^{m-1}}{C_{det}} F(t) \right)^{\frac{1}{2}} \\ &= R - \left(\frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(\omega)} dr \right)^{\frac{1}{2}}. \end{aligned}$$

Spatially distributed noise

- Recall:

$$\partial_t Y(t,x) = e^{\mu(t,x)} \Delta \left(e^{-\mu(t,x)} Y(t,x) \right)^m .$$

- freeze coefficients in space:

$$\partial_t Y(t,x) \approx e^{\mu(t,x_0)} \Delta \left(e^{-\mu(t,x_0)} Y(t,x) \right)^m ,$$

  on small balls $B_r(x_0)$.
- freeze coefficients in time:

$$\partial_t Y(t,x) \approx \Delta Y(t,x)^m ,$$

  for small times $t \approx 0$.

Hole-filling

- *Finite speed of hole-filling:*
  Let X_0 vanish in $B_R(x_0)$. Then X_t vanishes in $B_{R_{stoch(t,\omega)}}(x_0)$ with

$$R_{stoch}(t,\omega) = R - \left(\frac{H^{m-1}}{C_{det}} F(t,\omega)\right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega)$$
$$= R - \left(\frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(x_0,\omega)} dr\right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega),$$

  with $\lim_{R\downarrow 0} C_R = 1$.
- For $R \approx 0$ we recover the optimal rate from the spatially homogeneous case with $\mu_r \equiv \mu_r(\xi_0)$.

Hole-filling

- *Finite speed of hole-filling:*
  Let X_0 vanish in $B_R(x_0)$. Then X_t vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$R_{stoch}(t,\omega) = R - \left(\frac{H^{m-1}}{C_{det}} t \right)^{\frac{1}{2}} \sqrt{C_t(\omega)}.$$

  with $\lim_{t\downarrow 0} C_t = 1$.
- For $t \approx 0$ we recover the optimal rate from the deterministic case.

Finite speed of propagation

- *Finite speed of propagation:*
  Let X be an essentially bounded, non-negative solution to (SPME). Then,

$$supp(X_t) \subseteq B_{\sqrt{t}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}}\sqrt{C_t(\omega)}}(\mathrm{supp}(X_0)), \quad \forall t \in [0,T],$$

with $C_t \to 1$ for $t \to 0$.

Random attractors

Random attractors

Random attractors

- PME perturbed by linear multiplicative noise

$$dX_t = (\Delta(X_t^m) + \lambda X_t)\,dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}. \tag{2}$$

- The random attractor for (2) has infinite fractal dimension iff $\lambda > 0$.
- Cannot proceed via linearization
- Unstable manifold at 0:

$$\mathscr{M}^+(0,\omega) := \{u_0 \in X \mid \exists u : (-\infty, 0] \to X,\ \text{such that } \varphi(t;\theta_{-t}\omega)u(-t) = u_0 \\ \text{for all } t \geq 0 \text{ and } \|u(t)\|_{L^\infty(\mathscr{O})} \to 0 \text{ for } t \to -\infty\}.$$

Thanks

Thanks!