Gradient flow structures and large deviations for porous media equations

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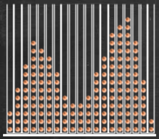
joint with Daniel Heydecker [CPAM 2025] and Ben Fehrman [Invent. Math. 2023].

Stochastic partial differential equations

Microscopic scale: Particles

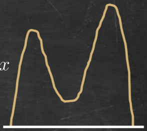


Gridsize = $\frac{1}{N}$

zero range process

$$\mu^N(x,t) := \frac{1}{N}\sum_k \delta_{\frac{k}{N}}(x)\eta(k,tN^2) \xrightarrow{N\to\infty} \bar\rho(t)\,dx$$

Macroscopic scale: PDEs

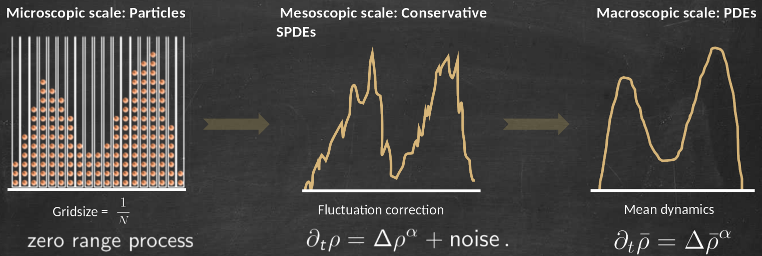


Mean dynamics

$$\partial_t \bar\rho = \Delta \bar\rho^\alpha$$

Replacing microscopic stochastic particle system by PDE leads to loss of information:

- Fluctuations, rare events / large deviations?
- Model / Approximation error: $\mu^N = \bar\rho + O(N^{-\frac{1}{2}})$.



Stochastic porous medium equation, $\alpha \geq 1$,

$$\partial_t \rho = \Delta \rho^\alpha + \underbrace{\text{noise}}_{?}\,.$$

Rewrite the PME as a gradient flow

$$\partial_t \bar\rho = \Delta \bar\rho^\alpha =_? -\nabla_{\mathcal{M}} \mathcal{H}(\bar\rho) = -M(\bar\rho)\frac{D\mathcal{H}}{D\rho}(\bar\rho),$$

where $M(\bar\rho)$ the inverse Riemannian tensor, $\mathcal{H}$ some entropy. Choose noise so that $\mu(d\rho) = \frac{1}{Z} e^{-\mathcal{H}(\rho)} d\rho$ becomes an invariant measure, i.e.

$$\partial_t \rho = -M(\rho)\frac{D\mathcal{H}}{D\rho}(\rho) + M^{\frac{1}{2}}(\rho) \diamond \xi.$$

Different gradient flow structures lead to different SPDEs.

Gradient flows for PME:

Brezis ['71]: $\mathcal{M} = H^{-1}$, $M(\bar\rho) = -\Delta$, $\mathcal{H}(\bar\rho) = \int \bar\rho^{\alpha+1}$,

$$\partial_t \bar\rho = \nabla \cdot (\nabla \bar\rho^\alpha).$$

Otto ['01]: $\mathcal{M} = \mathcal{P}(\mathbb{T}^d)$, $M(\bar\rho) = -\nabla \cdot (\bar\rho \nabla \cdot)$, $\mathcal{H}(\bar\rho) = \int \bar\rho^\alpha$ pressure,

$$\partial_t \bar\rho = \nabla \cdot (\bar\rho \nabla \bar\rho^{\alpha-1}).$$

"Thermodynamic metric": $\mathcal{M} = \mathcal{P}(\mathbb{T}^d)$, $M(\bar\rho) = -\nabla \cdot (\bar\rho^\alpha \nabla \cdot)$, $\mathcal{H}(\bar\rho)$ Boltzmann entropy,

$$\partial_t \bar\rho = \nabla \cdot (\bar\rho^\alpha \nabla \log(\bar\rho)).$$

Entropy dissipation inequality (EDI): Consider

$$\partial_t \bar\rho = -\nabla_{\mathcal{M}} \mathcal{H}(\bar\rho). \qquad (\star)$$

Note

$$\partial_t \mathcal{H}(\bar\rho) = -(\partial_t \bar\rho, \frac{D\mathcal{H}}{D\rho})_{M(\bar\rho)} \geq -|\partial_t \bar\rho|_{M(\bar\rho)} |\frac{D\mathcal{H}}{D\rho}|_{M(\bar\rho)} \geq -\frac{1}{2}|\frac{D\mathcal{H}}{D\rho}|^2_{M(\bar\rho)} - \frac{1}{2}|\partial_t \rho|^2_{M(\bar\rho)}$$

with equality iff $\bar\rho$ solves $(\star)$. $((v,w) = -\frac{1}{2}|v|^2 - \frac{1}{2}|w|^2$ iff $v = -w)$Integrate

$$\mathcal{I}(\bar\rho) = \mathcal{H}(\bar\rho(T)) - \mathcal{H}(\bar\rho(0)) + \frac{1}{2}\int_0^T |\frac{D\mathcal{H}}{D\rho}|^2_{M(\bar\rho)} + \frac{1}{2}\int_0^T |\partial_t \bar\rho|^2_{M(\bar\rho)} \geq 0$$

Consequence:$\bar\rho$ is a gradient flow for $(\star)$ iff

$$\bar\rho = \mathrm{argmin}_\rho \mathcal{I}(\rho)$$

Gradient flows and large deviations

Consider a particle system μ^N with hydrodynamic limit $\mu^N \to \bar{\rho}$ with

$$\partial_t \bar{\rho} = \Delta \bar{\rho}^\alpha.$$

Rare events are the (im-)probability to observe a fluctuation ρ:

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\mathcal{J}(\rho)} \quad N \text{ large},$$

for some rate function $\mathcal{J} \geq 0$.

Then, the typical behavior $\bar{\rho}$ is characterized as minimum of the rate function

$$\bar{\rho} = \operatorname{argmin}_\rho \mathcal{J}(\rho).$$

This links large deviations (determined by the particle system) with the EDI formulation of gradient flows.

Caveat, would still have to be able to rewrite the rate function $\mathcal{I}$ in EDI form, that is

$$\mathcal{J}(\bar{\rho}) = \mathcal{H}(\bar{\rho}(T)) - \mathcal{H}(\bar{\rho}(0)) + \frac{1}{2}\int_0^T |\frac{D\mathcal{H}}{D\rho}|^2_{M(\bar{\rho})} + \frac{1}{2}\int_0^T |\partial_t \bar{\rho}|^2_{M(\bar{\rho})}$$

Program

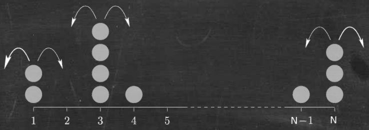
- Obtain porous medium equation as hydrodynamic limit
- Prove a large deviation principle
- Prove the EDI form of the rate function

The porous medium equation as a hydrodynamic limit

Can we obtain the PME as a limit of a (stochastic) particle system?

E.g. [Suzuki, Ushiyama; 1993], [Ekhaus, Seppäläinen; 1996], [Oelschläger; 1990], [Gonçalves, Landim, Toninelli; 2009], [Gonçalves, Nahum, Simon; 2023]

The zero range process:



Local jump rate function $g(\eta) = \eta^\alpha : \mathbb{N}_0 \to \mathbb{R}_0^+$, $\alpha > 1$.

Generator

$$L_N F(\eta)(x) := \frac{1}{2} \sum_{x \sim y \in \mathbb{T}_N^d} \eta^\alpha(x)(F(\eta^{x,y}) - F(\eta)).$$

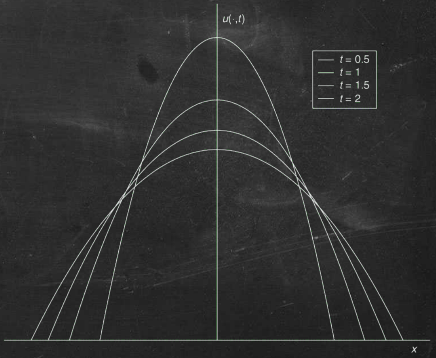
Hydrodynamic limit Empirical density field: Rescaling particle sizes by χ_N

$$\mu^N(x,t) := \chi_N\Big(\frac{1}{N}\sum_k \delta_k(x)\eta(k,t)\Big)(xN, t\frac{N^2}{\chi_N^{1-\alpha}}).$$

PME

$$\partial_t \bar{\rho} = \Delta \bar{\rho}^\alpha$$

Macroscopic: Barenblatt solution



Two difficulties:

- Superlinear growth of $g(\eta) = \eta^\alpha$, $\alpha > 1$. Possible concentration of mobility $(\eta^N(x))^\alpha$.
- Degeneracy of $g(\eta) = \eta^\alpha$ at $\eta = 0$. Now becomes visible with χ small. Dirichlet form degenerates.

As a result, the classical one-block, two-block approach to the superexponential replacement lemma is not applicable.

Solution: New microscopic, "pathwise" entropy-dissipation inequality

Theorem (Hydrodynamic limit, G., Heydecker, 2025)

Let $\rho_0 \in L^1_{\geq 0}(\mathbb{T}^d)$ *with finite entropy* $\mathcal{H}(\rho_0) = \int \rho_0 \log \rho_0 < \infty$, *initialize* $\eta_0^N \to \rho_0$. *Assume the scaling relation* $\chi_N^{1 \wedge \alpha/2} \leq CN^{-2}$. *Then*

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\, dx$$

in probability, where $\bar{\rho}$ *is the solution to*

$$\partial_t \bar{\rho} = \Delta \bar{\rho}^{\alpha}.$$

Large deviations around the porous medium equation?

Rate function

$$\mathcal{J}(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dxds : \; g \in L^2_{t,x}, \underbrace{\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)}_{\text{"skeleton equation"}} \right\}.$$

Theorem ([Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995])

For every $\mathcal{O} \subseteq D([0,T], \mathcal{M}_+)$ open set resp. we have

$$e^{-N \inf_{\rho \in \mathcal{O}} \overline{\mathcal{J}_{|A}}(\rho)} \lesssim \mathbb{P}[\mu^N \in \mathcal{O}] \lesssim e^{-N \inf_{\rho \in \bar{\mathcal{O}}} \mathcal{J}(\rho)}$$

where A is the set of nice fluctuations $\mu = \rho \, dx$ with ρ a solution to

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)$$

for some $\underline{g \in C^{1,3}_{t,x}}$. $\underline{\text{Problem:}}$ $\mathcal{J} = \overline{\mathcal{J}_{|A}}$?

One approach: Show well-posedness of

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g), \quad \text{with } g \in L^2_{t,x}.$$

Difficulties: No higher order regularity known. Only entropy-dissipation estimate.

Theorem (The skeleton equation, Fehrman, G. 2023)

Let $g \in L^2_{t,x}$, ρ_0 non-negative and $\int \rho_0 \log(\rho_0) dx < \infty$. There is a unique weak solution to

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

The map $g \mapsto \rho$, $L^2_{t,x} \to L^1_{t,x}$, is weak-strong continuous.

Theorem (LDP for zero range process, G., Heydecker, 2025)

The rescaled zero range process satisfies the full large deviations principle with speed $\frac{N^d}{\chi_N}$ and rate function

$$\mathcal{J}(\rho) = \inf \left\{ \|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \right\}.$$

Gradient flow structures for the porous medium equation

Can we rewrite $\mathcal{I}$ in EDI form?That is,

$$\mathcal{J}(\rho) =_? \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\int_0^T |\frac{D\mathcal{H}}{D\rho}|^2_{M(\rho)} + \frac{1}{2}\int_0^T |\partial_t \rho|^2_{M(\rho)}$$

Informally, we have

$$\begin{aligned}\mathcal{J}(\rho) =& \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2 dxds : \ g \in L^2_{t,x},\ \partial_t\rho = \Delta\rho^\alpha + \nabla\cdot(\rho^{\alpha/2} g)\right\}\\ =& \|\partial_t\rho - \Delta\rho^\alpha\|^2_{H^{-1}_{\rho^\alpha}} = \|\partial_t\rho\|^2 - 2(\partial_t\rho, -\nabla_{\mathcal{M}}\mathcal{H}(\rho^\alpha))_{H^{-1}_{\rho^\alpha}} + \|\Delta\rho^\alpha\|^2_{H^{-1}_{\rho^\alpha}}\\ =& \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\int_0^T \|\partial_t\rho\|^2_{\dot{H}^{-1}_{\rho^\alpha}} + \frac{1}{2}\|\Delta\rho^\alpha\|^2_{\dot{H}^{-1}_{\rho^\alpha}}.\end{aligned}$$

Define the action

$$\mathcal{A}(\rho) = \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t\rho + \nabla\cdot(\rho^{\alpha/2} g) = 0\} = \int_0^T \|\partial_t\rho\|^2_{\dot{H}^{-1}_{\rho^\alpha}}.$$

In conclusion, the gradient flow picture suggests the energy identity

$$\mathcal{J}(\rho) = \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\mathcal{A}(\rho) + \frac{1}{2}\int_0^T \|\rho^{\alpha/2}\|^2_{\dot{H}^1}.$$

Theorem (Entropy dissipation equality, G., Heydecker, 2025)

Let $D_\alpha(\rho) < \infty$, $\mathcal{H}(\rho_0) < \infty$, $u_0 > 0$. *Then*

$$\mathcal{J}(\rho) = \mathcal{H}_{u_0}(\rho_T) - \mathcal{H}_{u_0}(\rho_0) + \frac{1}{2}\mathcal{A}(\rho) + \frac{1}{2}\int_0^T \|\rho^{\alpha/2}(s)\|_{\dot{H}^1}^2 \, ds.$$

If ρ *is a solution to the PME, we have the energy equality*

$$0 = \mathcal{H}_{u_0}(\rho_T) - \mathcal{H}_{u_0}(\rho_0) + \int_0^T \|\rho^{\alpha/2}(s)\|_{\dot{H}^1}^2 \, ds.$$

Sketch of the proof

In equilibrium, detailed balance $\implies$ $(\mathcal{T}\eta^N_\bullet)_t := \eta^N_{T-t-}$ has the same law as the original process.

Contraction principle and uniqueness of rate functions:

$$\mathcal{I}(\mathcal{T}\rho) = \mathcal{I}(\rho)$$

for all ρ. Analyse identity without assuming any more regularity on ρ than necessary.

A new look at properties of the skeleton equation

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\alpha/2} g)$$

Construction of g_r shows how *anti*dissipative effects can arise, since

$$\partial_t \rho_r = -\Delta \rho_r^\alpha + \nabla \cdot (\rho_r^{\alpha/2} \mathcal{T} g)$$
$$= \Delta \rho_r^\alpha - \nabla \cdot \rho_r^{\frac{\alpha}{2}} (g_r).$$

Hence why L^p_x estimates had to be false: trajectories with $\rho_0 \notin L^p_x$, $\rho_T \in C^\infty_x$ give reversal $\rho_0 \in C^\infty_x$ but $\rho_T \notin L^p_x$.

References:

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Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift.

B. Gess and D. Heydecker.
The Porous Medium Equation: Large Deviations and Gradient Flow with Degenerate and Unbounded Diffusion.