Optimal regularity for the (nonlocal) anisotropic porous medium equation

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based on [G.; JEMS 2021], joint with Jonas Sauer, Eitan Tadmor [G., Sauer, Tadmor; Analysis & PDE, 2020], and joint with Jonas Sauer [G., Sauer; arxiv, 2023]

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Consider nonlocal, anisotropic degenerate PDE

$$\partial_t u = \mathcal{L}(u) \text{ on } (0,T) \times \mathbb{R}^d$$

with $u(0) = u_0 \in L^1(\mathbb{R}^d)$, where

$$\begin{aligned}\mathcal{L}(u)(x) &= \int_y \left(\Phi(u(x-y), x-y, y) - \Phi(u(x), x, y)\right) \nu(dy) \\ &= \int_y \left(\Phi(u(z), z, x-z) - \Phi(u(x), x, x-z)\right) d\nu(x-z)\end{aligned}$$

for some nonlinear function $\Phi(u, x, y)$ and measure ν.Special cases:

- Spatially homogeneous

$$\mathcal{L}(u) = \int_y \left(\Phi(u(x-y), y) - \Phi(u(x), y)\right) \nu(dy).$$

- Isotropic

$$\mathcal{L}(u) = \int_y \left(\Phi(u(x-y)) - \Phi(u(x))\right) \nu(dy) = \mathcal{L}(\Phi(u)).$$

- Local: Porous medium equation, $m \geq 1$,

$$\mathcal{L}(u) = \Delta\Phi(u) = \Delta\left(|u|^{m-1}u\right).$$

Aim: Optimal regularity of solutions in (fractional) Sobolev spaces.

Application/Derivation: Interacting (local) diffusions

Consider N particles X_t^i moving randomly according to a Brownian motion. Without interaction would get

$$dX_t^i = \sigma(X_t^i)dW_t^i \quad i = 1 \dots N.$$

Diffusion depends on the (local) empirical density of partices $\mu^N := \frac{1}{N}\sum_{j=1}^{N}\delta_{X^j}$. E.g. crowd-avoidance. Get

$$dX_t^i = \sigma(\mu^N, X_t^i)dW_t^i \quad i = 1 \dots N.$$

Propagation of chaos: Letting $N \to \infty$ we get $X^i \to X$ with

$$dX_t = \sigma(\mu_t, X_t) dW_t.$$

and $\mu = \mathcal{L}(X)$.
Fokker-Planck equation:

$$\partial_t \mu(x) = \sum_{i,j} \partial_{ij}(a_{ij}(\mu, x)\mu(x))$$

with $a = \sigma\sigma^*$.
Assume that particle X^i depends only on the number of particles X^j that are sufficiently close, get

$$\partial_t \mu(x) = \sum_{i,j} \partial^2_{ij}(a_{ij}((K^\varepsilon * \mu)(x))\mu(x))$$

Localized interaction/moderate interaction: $K^\varepsilon \to \delta$

$$\partial_t \mu(x) = \sum_{i,j} \partial^2_{ij}(a_{ij}(\mu(x))\mu(x))$$

Special, anisotropic case: $a_{ij}(\mu) = \frac{1}{m_i}\mu^{m_i - 1}\delta_{j=i}$,

$$\partial_t \mu(x) = \sum \partial^2_{ii}(\frac{1}{m_i}\mu^{m_i}(x))$$

Optimal regularity for the (local, homogeneous) porous medium equation

- Scaling and special solutions -

Note, for $m \geq 1$, $u^{[m]} := |u|^m \mathrm{sgn}(u)$,

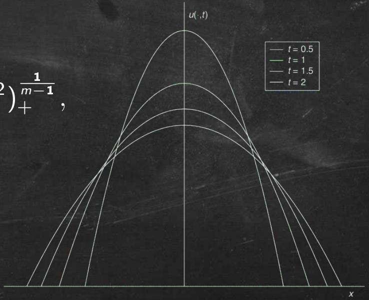
$$\partial_t u = \frac{1}{m}\Delta u^{[m]} = \mathrm{div}(|u|^{m-1}\nabla u)$$
$$= |u|^{m-1}\Delta u + (m-1)u^{[m-2]}|\nabla u|^2.$$

Barenblatt solution:

$$u_{BB}(x,t) = t^{-\alpha}F(xt^{-\beta}) = t^{-\alpha}(C - k|xt^{-\alpha/d}|^2)_+^{\frac{1}{m-1}},$$

where $\alpha = \frac{d}{d(m-1)+2}$, $k = \frac{(m-1)\alpha}{2md}$. We observe that

$$lim_{t\downarrow 0} u_{BB}(x,t) = M\delta_0(x)$$



What is the optimal order of differentiability?

Lemma

Assume that for some $s \geq 0$, $p \geq 1$, $C \geq 0$ we have

$$\|u\|^p_{L^p([0,T];\dot{W}^{s,p}(\mathbb{R}^d_x))} \leq C\|u_0\|_{L^1(\mathbb{R}^d_x)},$$

for all solutions u to PME. Then, necessarily $p \leq m$ and $s \leq \frac{2}{m}$.

Use scale invariances:

$$\tilde{u}(t,x) := u(\eta t, x)\eta^{\frac{1}{m-1}}, \quad \tilde{u}(t,x) := u(t, \eta x)\eta^{-\frac{2}{m-1}}.$$

Example

Consider the Barenblatt solution

$$u_{BB}(t,x) = t^{-\alpha}(C - k|xt^{-\beta}|^2)_+^{\frac{1}{m-1}}.$$

Then

$$u_{BB} \in L^m([0,T]; \dot{W}^{s,m}(\mathbb{R}^d_x))$$

implies $s < \frac{2}{m}$.

Use $u_{BB}(t,x) = t^{-\alpha}F(xt^{-\beta})$.

Existing regularity results

- Continuity

  Caffarelli-Friedman 1979, Sacks 1983, Cafarelli-Evans 1983, DiBenedetto 1983, Ziemer 1982
- Hölder continuity: α-Hölder continuity with $\alpha = \frac{1}{m} \in (0, 1)$.

  Caffarelli-Friedman 1980, DiBenedetto-Friedman 1985, Bögelein, Duzaar, Gianazza 2014
- Regularity of the open interface

  Caffarelli-Friedman 1980, Caffarelli-Vazquez-Wolansky 1987, Caffarelli-Wolanski 1990, Daskalopoulos-Hamilton 1998, Koch 1999
- Eventual C^∞ regularity

  Aronson-Vázquez 1987, Kienzler-Koch-Vazquez 2016
- Regularity of the pressure or powers of the solution

  Koch 1999, Gianazza-Schwarzacher 2016
- Time regularity (vanishing force)

  Aronson-Bénilan 1979, Crandall-Pazy-Tartar 1979, Bénilan-Crandall 1981, Crandall-Pierre 1982
- Regularity in Sobolev spaces

  [Lions-Perthame-Tadmor, JAMS, 1994], [Ebmeyer, JMAA, 2005], [Tadmor-Tao, CPAM, 2007]

Consider

$$\partial_t u = \frac{1}{m}\Delta u^{[m]}.$$

Let $\dot{\mathcal{N}}^{s,p}$ be the homogeneous Nikolskii space ($\dot{\mathcal{N}}^{s,p} = \dot{B}^s_{p,\infty}$).

Theorem (Tadmor, Tao; CPAM 2007, Ebmeyer; JMAA 2005)

Let $u_0 \in L^2(\mathbb{R}^d_x)$, $m \geq 1$. Then

$$\|u\|^{m+1}_{L^{m+1}([0,T];\dot{\mathcal{N}}^{\frac{2}{m+1},m+1}(\mathbb{R}^d_x))} \leq C_m \|u_0\|^2_{L^2_x}.$$

Note: $\frac{2}{m+1} \leq 1$, which is inconsistent with the linear case ($m = 1$) and with the optimal regularity of the Barenblatt solution.

By (soft) energy methods may be improved to:

Theorem (G., JEMS, 2021)

Let $\varepsilon > 0$, $m \geq 2$ and $u_0 \in L^{1+\varepsilon}(\mathbb{R}^d_x)$. Then

$$\|u\|^{m+\varepsilon}_{L^{m+\varepsilon}([0,T];\dot{\mathcal{N}}^{\frac{2}{m+\varepsilon},m+\varepsilon}(\mathbb{R}^d_x))} \leq C_{\varepsilon,m} \|u_0\|^{1+\varepsilon}_{L^{1+\varepsilon}_x}.$$

Note: optimal regularity for the Barenblatt solution, but $m \geq 2$ implies $\frac{2}{m+\varepsilon} < 1$.
Problem: How to get to more than one derivative?

Optimal regularity for the porous medium equation

Consider

$$\partial_t u = \frac{1}{m}\Delta u^{[m]} \quad \text{on } (0,T)\times\mathbb{R}^d_x \tag{PME}$$

with $u_0 \in L^1(\mathbb{R}^d_x)$.

Theorem (G., JEMS, 2021)

Let $\varepsilon > 0$, $u_0 \in L^{1+\varepsilon}(\mathbb{R}^d_x)$, $m \geq 1$. Then, for all

$$s \in [0, \frac{2}{m}), \quad p \in [1, m)$$

we have

$$u \in L^p([0,T]; \dot{W}^{s,p}_{loc}(\mathbb{R}^d_x)).$$

In addition, for all $\mathcal{O} \subset\subset \mathbb{R}^d$ there is a constant $C = C(m,p,s,T,\mathcal{O})$ such that

$$\|u\|_{L^p([0,T];\dot{W}^{s,p}(\mathcal{O}))} \leq C\left(\|u_0\|^2_{L^1_x} + 1\right).$$

The kinetic form [Lions, Perthame, Tadmor 1994], [Chen, Perthame; 2003]:For S regular, we have

$$\begin{aligned}\partial_t S(u) &= S'(u)\partial_t u = S'(u)\frac{1}{m}\Delta u^{[m]} \\ &= S'(u)\nabla\cdot(|u|^{m-1}\nabla u) \\ &= \nabla\cdot(S'(u)|u|^{m-1}\nabla u) - \underbrace{S''(u)\nabla u|u|^{m-1}\nabla u}_{\approx S''(u)|\nabla u^{\frac{m+1}{2}}|^2}\end{aligned}$$

Choosing $S_v(u) := (u-v)_+$ we get

$$\partial_t(u-v)_+ = \nabla\cdot(1_{u\geq v}|u|^{m-1}\nabla u) - \delta_{u=v}|\nabla u^{\frac{m+1}{2}}|^2$$

Taking the derivative in v yields

$$\begin{aligned}-\partial_t 1_{v<u(t,x)} &= -\nabla\cdot(\delta_{u(t,x)=v}|u|^{m-1}\nabla u) - \partial_v(\delta_{u(t,x)=v}|\nabla u^{\frac{m+1}{2}}|^2) \\ &= -\nabla\cdot(\delta_{u(t,x)=v}|v|^{m-1}\nabla u) - \partial_v(\delta_{u(t,x)=v}|\nabla u^{\frac{m+1}{2}}|^2) \\ &= -|v|^{m-1}\nabla\cdot(\nabla 1_{v<u(t,x)}) - \partial_v(\underbrace{\delta_{u(t,x)=v}|\nabla u^{\frac{m+1}{2}}|^2}_{=:q \text{ "entropy dissipation measure"}})\end{aligned}$$

Hence, with $\chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}$ we get the kinetic form

$$\partial_t\chi = |v|^{m-1}\Delta_x\chi + \partial_v q.$$

The isotropic case: Recall

$$\partial_t \chi = |v|^{m-1}\Delta_x \chi + \partial_v q \text{ on } (0,T)\times\mathbb{R}^d_x\times\mathbb{R}_v,$$

Variation of constants/Duhamel

$$\chi(t,x,v) = e^{-|v|^{m-1}t\Delta}\chi_0(x,v) + \int_0^t e^{-|v|^{m-1}(t-r)\Delta}\partial_v q(r,x,v)dr.$$

Decompose u in degenerate and non-degenerate part, using that $\chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}$,

$$u(t,x) = \int_v \chi(u(t,x),v) = \underbrace{\int_{|v|\le\delta}\chi(u(t,x),v)}_{u^0(t,x)} + \underbrace{\int_{|v|\ge\delta}\chi(u(t,x),v)}_{u^1(t,x)}.$$

Note:

$$u^1(t,x) = \int_{|v|\ge\delta} e^{-|v|^{m-1}t\Delta}\chi_0(x,v) + \int_0^t\int_{|v|\ge\delta} e^{-|v|^{m-1}\Delta(t-r)}\partial_v q(r,x,v).$$

Trivial estimate:

$$\|u^0\|_{L^\infty_{t,x}} = \left\| \int_{|v|\le\delta} \chi(u(t,x),v) \right\|_{L^\infty_{t,x}} \lesssim \delta.$$

Recall:

$$u^1(t,x) = \int_{|v|\ge\delta} e^{-|v|^{m-1}t\Delta}\chi_0(x,v) + \int_0^t \int_{|v|\ge\delta} e^{-|v|^{m-1}\Delta(t-r)}\partial_v q(r,x,v).$$

Heat kernel estimates: Up to "epsilons" (!)

$$\|u^1\|_{L^1_t H^{2,1}_x} \lesssim \delta^{-1-(m-1)}\|q\|_{\mathcal{M}_{t,x,v}}.$$

Test case: $m=1$, get $u \in L^1_t W^{1,1}_x$.
Singular moments

$$\|u^1\|_{L^1_t H^{2,1}_x} \lesssim \delta^{-m-1}\||v|^{-1}q\|_{\mathcal{M}_{t,x,v}}.$$

and note

$$|v|^{-1}q = |v|^{-1}\delta_{u(t,x)=v}|\nabla u^{\frac{m+1}{2}}|^2 \approx \delta_{u(t,x)=v}|\nabla u^{\frac{m}{2}}|^2$$

Fisher-entropy dissipation.

Recall, up to "epsilons" (!):

$$u = u^0 + u^1,$$

$$\|u^0\|_{L^\infty_{t,x}} = \left\| \int_{|v|\le\delta} \chi(u(t,x),v) \right\|_{L^\infty_{t,x}} \lesssim \delta,$$

$$\|u^1\|_{L^1_t H^{2,1}_x} \lesssim \delta^{-(m-1)} \||v|^{-1} q\|_{\mathcal{M}_{t,x,v}}.$$

Real interpolation: K-functional

$$K(z,u) = \inf\left\{ \left\|u^0\right\|_{L^\infty_{t,x}} + z\left\|u^1\right\|_{L^1_t H^{2,1}_x} \ : \ u = u^0 + u^1 \right\}.$$

Then

$$\begin{aligned} \|u\|_{(L^\infty_{t,x}, L^1_t H^{2,1}_x)_{\theta,\infty}} &= \|z^{-\theta} K(z,u)\|_\infty \\ &\lesssim \|z^{-\theta}(\delta + z\delta^{-(m-1)})\|_\infty \end{aligned}$$

Equilibrating z, δ : Require $\delta = z\delta^{-(m-1)}$ which implies $\delta = z^{\frac{1}{m}}$. Get

$$\|u\|_{(L^\infty_{t,x}, L^1_t H^{2,1}_x)_{\theta,\infty}} \lesssim \|z^{-\theta} z^{\frac{1}{m}}\|_\infty.$$

Hence choose $\theta = \frac{1}{m}$,and, up to "epsilons",

$$\|u\|_{L^m_t W^{\frac{2}{m},m}_x} = \|u\|_{(L^\infty_{t,x}, L^1_t H^{2,1}_x)_{\frac{1}{m},\infty}} < \infty.$$

The anisotropic case: Above, we *essentally* used the isotropy. Anisotropic case needs a different arguments. Indeed, for anisotropic PME, with $m_1 \leq ... \leq m_d$,

$$\partial_t u = \sum_i \partial^2_{x_i x_i} \frac{1}{m_i} u^{[m_i]} \qquad (\star)$$

kinetic form becomes

$$\partial_t \chi = \sum_i |v|^{m_i - 1} \partial^2_{x_i x_i} \chi + \partial_v q$$

and Fourier transformation of $(\star)$ in time and space (modulo cut-off in time)

$$\underbrace{i\tau\hat{\chi} - \sum_i |v|^{m_i-1} \xi_i^2 \hat{\chi}}_{=:\mathcal{L}(i\tau,\xi,v)\hat{\chi}} = \partial_v \hat{q}.$$

That is

$$\hat{\chi} = \mathcal{L}^{-1}(i\tau, \xi, v) \partial_v \hat{q}.$$

Idea: Micro-local decomposition of Fourier-space depending on the degeneracy in $|v|$.
Aim: Micro-local decomposition is chosen so that all regularity is on $\tilde{u}^0$, while $\tilde{u}^1$ is only $L^1_{t,x}$:

$$u \in (\underbrace{L^r_t H^{2\alpha,r}_x}_{\ni \tilde{u}^0}, \underbrace{L^1_{t,x}}_{\ni \tilde{u}^1})_{\theta,\infty} \subseteq L^{m-\varepsilon}_t W^{\frac{2}{m}-\varepsilon, m-\varepsilon}_x.$$

Informally,

$$\hat{\chi} = \mathcal{L}^{-1}(i\tau, \xi, v)\partial_v \hat{q} = \frac{1}{i\tau - \sum_i |v|^{m_i - 1}\xi_i^2}\partial_v \hat{q}.$$

Gain regularity, depending on the degeneracy of the operator $\mathcal{L}(i\tau, \xi, v)$. With the dyadic partition of unity

$$\phi_0(\xi) + \sum_{j \geq 1} \phi_1(2^{-j}\xi) = 1$$

construct the micro-local decomposition of χ by

$$\hat{\chi} = \underbrace{\phi_0(\frac{\mathcal{L}(i\tau, \xi, v)}{\delta})\hat{\chi}}_{\chi^0} + \sum_{j \geq 1} \underbrace{\phi_1(\frac{\mathcal{L}(i\tau, \xi, v)}{2^j \delta})\hat{\chi}}_{\chi_j^1}$$

and get

$$\hat{u} = \underbrace{\int_v \phi_0(\frac{\mathcal{L}(i\tau, \xi, v)}{\delta})\hat{\chi}}_{u^0} + \underbrace{\int_v \sum_{j \geq 1} \phi_1(\frac{\mathcal{L}(i\tau, \xi, v)}{2^j \delta})\hat{\chi}}_{u^1}$$

Paley-Littlewood decomposition (in space) to work on fixed blocks of Fourier modes: $\hat{\chi}_J$, $\hat{u}_J$.

On non-degenerate parts use the equation ($\hat{\chi} = \frac{1}{\mathcal{L}(i\tau,\xi,v)}\partial_v \hat{q}$) and velocity-average.

Establish multiplier estimates to control regularity of u_J^0 on Paley-Littlewood blocks,

$$\|u_J^0\|_{L^p_{t,x}} = \| \int_v \phi_0(\frac{\mathcal{L}(i\tau,\xi,v)}{\delta})\chi_J \, dv\|_{L^p_{t,x}} \lesssim^{!?} \delta^\alpha J^{-2\alpha}\|\chi\|_{L^p_{t,x}H^{\sigma,p}_v}.$$

Obstacles:

1. Bootstrapping: Established methods rely on bootstrapping, i.e. assuming that $u \in W^{\alpha,1}_x$ for some α use that $\chi(u) = 1_{v<u(t,x)} - 1_{v<0} \in W^{\alpha,1}_{x,v}$. But: This is true for $\alpha \leq 1$ only!
2. Established methods can only make use of the fact that q has finite mass. This necessarily leads to sub-optimal estimates.
   –> Solution: Use that q allows singular moments $\int |v|^{-1+} dq < \infty$.

Theorem (G., JEMS, 2021)

Let $u_0 \in L^\infty(\mathbb{R}^d_x)$, $m_j \geq 1$, $j = 1, \dots, d$ and let u be the entropy solution to

$$\partial_t u = \sum_{i=1}^{d} \partial^2_{x_i x_i} \frac{1}{m_i} u^{[m_i]} \quad \text{on } (0,T) \times \mathbb{R}^d.$$

We set $\underline{m} = \min(\{m_j : j = 1, \dots, d\})$, $\overline{m} = \max(\{m_j : j = 1, \dots, d\})$. Then, for all

$$s \in \Big[1, \frac{2}{\overline{m}} \left(\frac{\underline{m} - 1}{\overline{m} - 1} \right) \Big), \quad p \in \Big[1, \frac{2\overline{m}}{1 + \overline{m}} \Big),$$

and $\mathcal{O} \subset\subset \mathbb{R}^d$ there is a constant $C \geq 0$ such that

$$\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \leq C \left(\|u_0\|^{1+}_{L^{1+}_x} + 1 \right).$$

Optimal regularity for nonlocal, inhomogeneous porous medium equation

Recall: (Local) Fokker-Planck equation

$$\partial_t \mu = \sum_{i,j} \partial_{ij}(a_{ij}(\mu, x)\mu).$$

How does a nonlocal generalization look like?

Application/Derivation: Consider N particles X_t^i moving randomly according to a Poisson random measure $N(dr, dy)$ on $[0, T] \times \mathbb{R}^d$, with intensity measure $n(dr, dy) := dr\, \nu(dy)$, and ν a Levy measure on $\mathbb{R}^d$. Without interaction would get

$$dX_t^i = \int_y g(y) N^i(dt, dy) \quad i = 1 \ldots N.$$

Diffusion depends on the (local) empirical density of partices $\mu^N := \frac{1}{N} \sum_{j=1, j \neq i}^{N} \delta_{X^j}$:

$$dX_t^i = \int_y g((\mu_{r-}^N * K^\varepsilon)(X_{r-}), y) N^i(dt, dy) \quad i = 1 \ldots N.$$

Propagation of chaos: Letting $N \to \infty$ we get

$$X_t = X_0 + \int_0^t \int_y g((\mu_{r-} * K^\varepsilon)(X_{r-}), y) \, N(dr, dy),$$

with $\mu = \mathcal{L}(X)$. Localized interaction/moderate interaction: $K^\varepsilon \to \delta$

$$X_t = X_0 + \int_0^t \int_y g(\mu_{r-}(X_{r-}), y) \, N(dr, dy).$$

Fokker-Planck equation:

$$\begin{aligned}
\partial_t < \mu(t), f > &= \partial_t \mathbb{E} f(X_t) \\
&=_{\text{Itô}} \int_x \int_y \mu(t,x) \left(f(x + g(\mu(t,x), x, y)) - f(x) \right) \nu(dy) \\
&= \int_x \int_y \mu(t,x) \left(f(x+y) - f(x) \right) g(\mu(t,x), x, \cdot)_* \nu(dy) \\
&= \int_x \int_y \mu(t,x) \left(f(x+y) - f(x) \right) \tilde{\Phi}(\mu(t,x), x, y) \, \nu(dy) \\
&=< \mathcal{L}(\mu(t)), f >
\end{aligned}$$

with

$$\tilde{\Phi}(\mu, x, y) := \frac{dg(\mu, x, \cdot)_* \nu}{d\nu(\cdot)}(y)$$

Hence, Fokker-Planck equation becomes a nonlocal, anisotropic, degenerate PDE

$$\partial_t \mu = \mathcal{L}(\mu) := \int_y \left(\Phi(\mu(t, x-y), x-y, y) - \Phi(\mu(t,x), x, y)\right) \nu(dy).$$

Special case

$$\partial_t \mu = \Delta^{a/2} \mu^m.$$

For every $m \in (1, \infty)$ and $a \in (0, 2)$ there is suitable bounded and Hölder continuous function $f : [0, \infty) \to \mathbb{R}$, such that

$$u_{bb}(t, x) := t^{-\alpha} f(|x| t^{-\beta}),$$

with $\alpha := \frac{d}{d(m-1)+a}$ and $\beta = \frac{\alpha}{d}$ is a self-similar solution [Vázquez, J. Eur. Math. Soc.; 2014].

Lemma

Let $m \in (1, \infty)$ and $a \in (0, 2)$. Then,

$$u_{BB} \in L^m(0, T; \dot{W}^{\sigma, m}(\mathbb{R}^d))$$

implies $\sigma < \frac{a}{m}$.

Recall

$$\partial_t u = \mathcal{L}(u) := \int_y \left(\Phi(u(t,x-y),x-y,y) - \Phi(u(t,x),x,y)\right)\nu(dy).$$

Informally, the kinetic form reads

$$\begin{aligned}\partial_t \chi - \int_{\mathbb{R}^d} (\Phi_v(v,x+y,y)\chi(x+y) - \Phi_v(v,x,y)\chi(x))\nu(dy) \\ = \partial_v\Big(\chi(x)\int_{\mathbb{R}^d}(\Phi(v,x+y,y) - \Phi(v,x,y))\nu(dy)\Big) + \partial_v q = \partial_v \tilde{q}.\end{aligned}$$

In Fourier space this becomes

$$\underbrace{(i\tau - \int_{\mathbb{R}^d}(e^{-iy\cdot\xi} - 1)\Phi_v(v,x,y)\tilde{\nu}(dy))}_{:=\mathcal{L}_x(i\tau,\xi,v)}\hat{\chi}(\tau,\xi) = \partial_v \hat{\tilde{q}},$$

where $\mathcal{L}_x(i\tau,\xi,v)$ is a pseudodifferential operator with symbol

$$p_v(x,\xi) := \int_{\mathbb{R}^d}(e^{iy\cdot\xi} - 1)\Phi_v(v,x,y)\nu(dy).$$

Challenges:

- Microlocal analysis of the kinetic operator

$$\mathcal{L}_x(i\tau, \xi, v) = \int_{\mathbb{R}^d} (e^{-iy\cdot\xi} - 1)\Phi_v(v, x, y)\nu(dy).$$

  Extension of the parametrix method "$\mathcal{L}_x^{-1}(i\tau, \xi, v) = ...$" for pseudodifferential operators providing precise bounds in terms of v .
- Since the parametrix is not an exact inverse, lower order terms need to be controlled. Careful interpolation and absorption argument.
- (Possibly non-unique) generalized kinetic solutions.

Assume, for some $a \in (0,2)$, $m \in (1,\infty)$

$$|\xi|^a \lambda_2(v) \geq Re\, p_v(x,\xi) \geq \lambda_1(v)|\xi|^a$$
$$|\{\lambda_1(v)^2 \lambda_2(v)^{-1} < \delta\}| \leq C\delta^{\frac{1}{m-1}}.$$

Theorem (G., Sauer, 2023)

Let $u_0 \in L^1(\mathbb{R}^d)$ and u be a generalized kinetic solution, $\eta \in C_c^\infty(\mathbb{R}_v)$ such that $\eta\chi \in L^1_{t,x,v}$, $|\eta| \leq 1$ and $\eta' \in L^1(\mathbb{R}_v)$, and write $u^\eta(t,x) := \int_{\mathbb{R}_v} \eta(v)\chi(t,x,v)\,dv$. Let $p \in (1,m)$ and define

$$\kappa_t := \frac{m-p}{p}\frac{1}{m-1}, \quad \kappa_x := \frac{p-1}{p}\frac{a}{m-1}.$$

Then, for all $\sigma_t \in [0,\kappa_t)$ and $\sigma_x \in [0,\kappa_x)$, $\eta \in C_c^\infty(\mathbb{R}_v)$ we have

$$u^\eta \in \dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d)).$$

Moreover, we have the estimate

$$\|u^\eta\|^p_{\dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d)).} \lesssim (1+\|\eta'\|_{L^1_v})(\|u_0\|^m_{L^1_x}+\|S\|^m_{L^1_{t,x}}+1)+\|\eta{\lambda_1}^2{\lambda_2}^{-1}f\|_{L^1_{t,x,v}}.$$

Special case

$$\partial_t u = \Delta^{a/2} u^m.$$

Corollary (Optimal space-time regularity for nonlocal PME, G., Sauer, 2023)

Let $u_0 \in L^1(\mathbb{R}^d)$, and assume $a \in (0,2)$, $m \in (1,\infty)$. Let $p \in (1,m]$ and

$$\kappa_t := \frac{m-p}{p}\frac{1}{m-1}, \quad \kappa_x := \frac{p-1}{p}\frac{a}{m-1}.$$

Then, for all $\sigma_t \in [0,\kappa_t) \cup 0$ and $\sigma_x \in [0,\kappa_x)$ we have

$$u \in \dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d))$$

and for all $\varepsilon > 0$ there holds

$$\|u\|^p_{\dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d))} \lesssim \|u_0\|^{m+\varepsilon}_{L^1_x} + 1,$$

where the implicit constant depends only on d, a, m, p and ε.

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