Well-posedness and regularization by noise for nonlinear PDE

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joint work with: Panagiotis E. Souganidis, Benoit Perthame, Paul Gassiat
[G., Souganidis; CMS, 2014], [G., Souganidis; CPAM, 2016],
[G., Perthame, Souganidis; SINUM, 2016], [Gassiat, G.; ongoing].

Outline

1. Introduction
2. Regularity of solutions to stochastic scalar conservation laws
3. Regularization by noise for stochastic Hamilton-Jacobi equations

Introduction

- General aim: Regularization or well-posedness by inclusion of stochastic perturbations
- Problem: A-priori unclear which form of noise to consider
- Additive noise, e.g.

$$du = \Delta u\, dt + f(u)\, dt + dW_t \quad \text{[Gyöngy, Pardoux; 1993]}$$

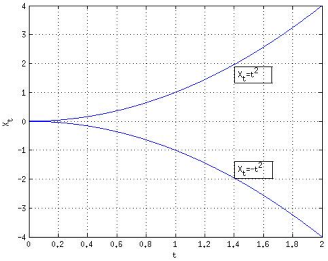
$$du + (u \cdot \nabla) u\, dt + \nabla p\, dt = \Delta u\, dt + dW_t \quad \text{[Flandoli, Romito; 2007].}$$

- More recent: Linear multiplicative noise

$$du + b(x) \cdot \nabla u\, dt = \nabla u \circ d\beta_t. \quad \text{[Flandoli, Gubinelli, Priola; 2010].}$$

Introduction

- We recall: Consider
$$du + b(x) \cdot \nabla u = 0 \qquad \text{(TE)}$$
for non-Lipschitz b (but, say, Hölder continuous). E.g. $b(x) = \operatorname{sgn}(x)\sqrt{|x|}$.
- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than $u(t) \in BV$ even if u_0 is smooth.
- Weak solutions are non-unique: e.g. $b(x) = \operatorname{sgn}(x)\sqrt{|x|}$

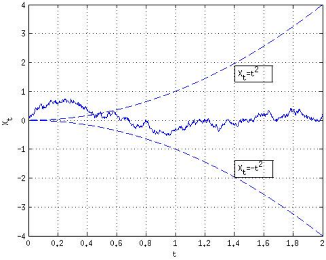


- Question: Can noise restore uniqueness or increase regularity?

Introduction

- Consider
$$du + b(x)\cdot\nabla u = \nabla u \circ d\beta_t. \qquad \text{(STE)}$$
- Two (related) effects: regularization by noise, well-posedness by noise.
- Regularization by noise [Flandoli, Fedrizzi; 2013]: If u_0 is smooth then $u(t)$ is smooth.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; 2010]: Weak solutions to (STE) are unique.

$$du + b(x)\cdot\nabla u = \nabla u \circ d\beta_t$$
$$b(x) = \mathrm{sgn}(x)\sqrt{|x|}$$



Introduction

- Left open: What about the nonlinear case, e.g. Burgers?
- Linear multiplicative noise does not help anymore:

$$du + \partial_x u^2\, dt = \partial_x u \circ d\beta_t.$$

- Then: $v(t,x) := u(t, x-\beta_t)$ is the unique solution to

$$\partial_t v + \partial_x v^2 = 0.$$

- Side-remark: Space-time linear multiplicative noise, that is

$$\sum_{k=1}^{\infty} \sigma_k(x) \cdot \nabla u \circ d\beta_t^k$$

with β^k independent Brownian motions seems more promising. More details: [Delarue, Flandoli, Vincenzi; 2014].

Regularity of solutions to stochastic SCL

Regularity of solutions to stochastic scalar conservation laws

Regularity of solutions to stochastic SCL

- Consider
$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta_j = 0, \qquad \text{(SSCL)}$$
on the torus $\mathbb{T}^N$, $A \in C^2$.
- Assume that the flux A is non-degenerate: I.e. there exist $\theta \in (0,1]$ and $C > 0$ such that, for all $\sigma \in S^{N-1}$, $z \in \mathbb{R}^N$ and $\varepsilon > 0$,
$$|\{\xi \in \mathbb{R} : |A'(\xi)\sigma - z| \le \varepsilon\}| \le C\varepsilon^{\theta}.$$
- e.g. A strictly convex.

Regularity of solutions to stochastic SCL

Theorem (G., Souganidis; CPAM, 2016)

Let u be the unique entropy solution to (SSCL). For all $\lambda \in (0, \frac{4\theta}{2\theta+3})$, $T>0$, there is a $C > 0$ such that

$$\mathbb{E}\int_0^T \|u(t)\|_{W^{\lambda,1}}\,dt \le C(1+\|u_0\|_2^2)$$

and, for all $\delta > 0$,

$$\sup_{t\ge\delta}\mathbb{E}\|u(t)\|_{W^{\lambda,1}} < \infty.$$

Regularization by nonlinear noise

- Consider the Burgers' equation

$$\partial_t u + \partial_x u^2 = 0 \quad \text{on } \mathbb{T}. \tag{B}$$

- Reminder: The kinetic form of (B) is obtained via $\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi)$ which solves

$$\partial_t \chi + \xi \partial_x \chi = \partial_\xi m$$

  for some non-negative, finite measure m.

- We consider quasi-solutions [De Lellis, Otto, Westdickenberg; 2003]: A weak solution u to (B) is a quasi-solution, if for some Radon measure m

$$\partial_t \chi + \xi \partial_x \chi = \partial_\xi m.$$

- Quasi-solutions to (B) are not unique.

(New) results for the stochastic case

- [De Lellis, Westdickenberg; 2003]: There is a quasi-solution to (B) such that

$$u(t) \notin W^{\lambda,1} \quad \text{for all } \lambda > \frac{1}{3}.$$

- Question: Does noise improve the situation?
- Consider the stochastic Burgers' equation

$$du + \partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T}. \tag{SB}$$

Theorem (G., Souganidis; CPAM, 2016)

Let $u \in L^\infty$ be a pathwise quasi-solution to (SB). Then, $t > 0$,

$$u(t) \in W^{\lambda,1} \quad \textit{for all } \lambda \in (0, \frac{4}{5}), \mathbb{P}\text{-a.s.}.$$

- Thus: quasi-solutions to (SB) are more regular than to (B), i.e. regularization by noise.

Idea of the proof

- By definition quasi-solutions satisfy

$$d\chi + \xi \partial_x \chi \circ d\beta_t = \partial_\xi m,$$

  for some finite Radon measure m.
- Change of variables gives

$$\chi(t,x,\xi) = \int_0^t \chi_0(x+\xi(\beta_t-\beta_s),\xi)ds + \int_0^t \partial_\xi m(s,x+\xi(\beta_t-\beta_s),\xi)ds.$$

- Averaging over velocity

$$u(t,x) = \int_0^t \int_\xi \chi_0(x+\xi(\beta_t-\beta_s),\xi)d\xi ds + \int_0^t \int_\xi \partial_\xi m(s,x+\xi(\beta_t-\beta_s),\xi)d\xi ds.$$

- The averaging effect appears since the velocity average in ξ contains averaging of the x-variable.

Idea of the proof

- Rigorously, this can be seen by Fourier transform, that is,

$$\hat{u}(t,n) = \int_0^t \int_\xi e^{-i\xi(\beta_t-\beta_s)n} \hat{\chi}_0(n,\xi)\, d\xi\, ds + \int_0^t \int_\xi e^{-i\xi(\beta_t-\beta_s)n} \partial_\xi \hat{m}(s,n,\xi)\, d\xi\, ds.$$

- The oscillatory integrals have a regularizing effect, both in ξ and in $\beta_t - \beta_s$.

Regularization by noise for stochastic Hamilton-Jacobi equations

Regularization by noise for stochastic Hamilton-Jacobi equations

Introduction

- Can we use nonlinear noise to regularize PDE?
- Model example: Porous medium equation

$$\partial_t w = \frac{1}{6}\partial_{xx} w^3, \quad \text{on } \mathbb{R}$$

  with smooth, compactly supported initial condition.
- Solutions are known to develop singularities:

$$\|\partial_x w(t)\|_{L^\infty} = \infty$$

  for all $t > 0$ large enough.
- Linear multiplicative noise does not help:

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \partial_x v \circ d\beta_t, \quad \text{on } \mathbb{R}.$$

  Then $w(t,x) = v(t, x - \sigma\beta_t)$.

Introduction

- Instead, consider, for $\sigma > 0$,

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \frac{1}{2}\partial_x v^2 \circ d\beta_t, \quad \text{on } \mathbb{R}. \tag{SPME}$$

- Note: If u is the viscosity solution to

$$du = \frac{1}{6}\partial_x(\partial_x u)^3\, dt + \sigma \frac{1}{2}(\partial_x u)^2 \circ d\beta_t, \quad \text{on } \mathbb{R},$$

then, $v = \partial_x u$ solves (SPME).

Setup

- General framework: Consider

$$du = F(t,x,u,Du,D^2u)\,dt + \frac{1}{2}|Du|^2 \circ d\beta_t, \quad \text{on } \mathbb{R}^N,$$

  and compare to the regularity of non-perturbed problem

$$dw = F(t,x,w,Dw,D^2w), \quad \text{on } \mathbb{R}^N.$$

- F satisfies the usual assumptions from the theory of stochastic viscosity solutions

Key result

- Control on the rate of loss of semiconcavity: There is a $V_F \in Lip_{loc}(\mathbb{R}_+ \setminus \{0\})$ such that, for $\ell_0 > 0$,

$$D^2 g \le \frac{Id}{\ell_0} \;\Rightarrow\; D^2(S_F(t,g)) \le \frac{Id}{\ell_0(t)},$$

where $t \mapsto S_F(t,g)$ denotes the solution to

$$\begin{aligned} \partial_t w + F(t,x,w,Dw,D^2 w) &= 0 \\ w(0) &= g \end{aligned}$$

and ℓ the solution to

$$\begin{aligned} \dot{\ell}(t) &= V_F(\ell(t)) \\ \ell(0) &= \ell_0. \end{aligned}$$

Key result

Theorem

Let u be the solution to

$$\partial_t u = F(t,x,u,Du,D^2u) + \tfrac{1}{2}|Du|^2 \circ d\beta_t,$$

with

$$D^2 u_0 \leq \frac{Id}{\ell_0} \quad \textit{some } \ell_0 \in [0,\infty).$$

Then for each $t \geq 0$, one has

$$D^2 u(t,\cdot) \leq \frac{Id}{L_t}, \tag{1}$$

where L is the maximal solution to

$$\begin{aligned} dL_t &= V_F(L_t)dt + d\beta_t, \ \textit{with } L_t \geq 0, \\ L_0 &= \ell_0. \end{aligned} \tag{2}$$

Model example

- Return to the model example

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \frac{1}{2}\partial_x v^2 \circ d\beta_t, \quad \text{on } \mathbb{R}.$$

- Determinstic case: $\|\partial_x w(t)\|_\infty = \infty$ for all $t > 0$ large enough.
- We have the sharp bound

$$\|\partial_x v(t)\|_\infty \leq \frac{1}{L^+(t) \wedge L^-(t)},$$

where $L^\pm$ solve

$$dL^\pm = -\frac{1}{L^\pm(t)}\, dt \pm \sigma d\beta_t, \text{ with } L^\pm_t \geq 0,$$
$$L^\pm(0) = \frac{1}{\|(\partial_x v_0)_\pm\|_\infty}.$$

Model example

- In conclusion,
  - If $\sigma^2 > 2$: For all $t \geq 0$, $\mathbb{P}$-a.s.

$$v(t) \in W^{1,\infty}$$

  - If $\sigma^2 \leq 2$: $\mathbb{P}$-a.s. for all $t > 0$ large enough

$$v(t) \notin W^{1,\infty}$$

Thanks

Thanks!