Gradient flow structures for porous media equations, large deviations, and multiscale analysis

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Pattern Formation, Energy Landscapes, and Scaling Laws - celebrating Felix Otto as a mentor, collaborator, and mathematician

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joint with Ben Fehrman [Invent. Math. 2023] and Daniel Heydecker [CPAM 2025], [Arxiv 2026]

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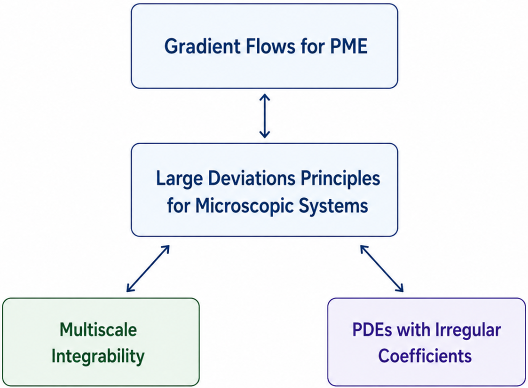

Gradient Flows for PME
Large Deviations Principles for Microscopic Systems
Multiscale Integrability
PDEs with Irregular Coefficients

Porous medium equation

$$\partial_t \rho = \Delta \rho^\alpha, \quad \alpha > 1.$$

Gradient flows for PME:

$$\partial_t \rho = -\nabla_{\mathcal{M}} \mathcal{H}(\rho) = -M(\rho) \frac{D\mathcal{H}}{D\rho}(\rho).$$

Brezis ['71]: $\mathcal{M} = H^{-1}$, $M(\rho) = -\Delta$, $\mathcal{H}(\rho) = \int \rho^{\alpha+1}$,

$$\partial_t \rho = \nabla \cdot (\nabla \rho^\alpha).$$

Otto ['01]: $\mathcal{M} = \mathcal{P}(\mathbb{T}^d)$, $M(\rho) = -\nabla \cdot (\rho \nabla \cdot)$, $\mathcal{H}(\rho) = \int \rho^\alpha$ pressure,

$$\partial_t \rho = \nabla \cdot (\rho \nabla \rho^{\alpha-1}).$$

"Thermodynamic metric": $\mathcal{M} = \mathcal{P}(\mathbb{T}^d)$, $M(\rho) = -\nabla \cdot (\rho^\alpha \nabla \cdot)$, $\mathcal{H}(\rho) = \mathcal{H}(\rho)$ Boltzmann entropy,

$$\partial_t \rho = \nabla \cdot (\rho^\alpha \nabla \log(\rho)).$$

What is the "right" gradient flow? How to select a gradient flow?

Entropy dissipation inequality (EDI): Note, for any differentiable $\bar{\rho}$,

$$\partial_t \mathcal{H}(\bar{\rho}) = (\partial_t \bar{\rho}, \frac{D\mathcal{H}}{D\rho})_{M(\bar{\rho})} \geq -|\partial_t \bar{\rho}|_{M(\bar{\rho})} |\frac{D\mathcal{H}}{D\rho}|_{M(\bar{\rho})} \geq -\frac{1}{2} |\frac{D\mathcal{H}}{D\rho}|^2_{M(\bar{\rho})} - \frac{1}{2} |\partial_t \bar{\rho}|^2_{M(\bar{\rho})}$$

with equality iff $\bar{\rho}$ solves

$$\partial_t \bar{\rho} = -\nabla_{\mathcal{M}} \mathcal{H}(\bar{\rho}).$$

Integrate time

$$\tilde{\mathcal{I}}(\bar{\rho}) := \mathcal{H}(\bar{\rho}(T)) - \mathcal{H}(\bar{\rho}(0)) + \frac{1}{2} \int_0^T |\frac{D\mathcal{H}}{D\rho}|^2_{M(\bar{\rho})} + \frac{1}{2} \int_0^T |\partial_t \bar{\rho}|^2_{M(\bar{\rho})} \geq 0$$

Consequence: $\bar{\rho}$ is a gradient flow for $(\star)$ iff

$$\bar{\rho} = \mathrm{argmin}_\rho \tilde{\mathcal{I}}(\rho)$$

Gradient flows and large deviations

Consider a stochastic particle system μ^N with hydrodynamic limit $\mu^N \to \bar{\rho}$ with

$$\partial_t \bar{\rho} = \Delta \bar{\rho}^\alpha.$$

Rare events are the (im-)probability to observe a fluctuation ρ:

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N^d \mathcal{I}(\rho)} \quad N \text{ large},$$

for some rate function $\mathcal{I} \geq 0$.

Then, the typical behavior $\bar{\rho}$ is characterized as minimum of the rate function

$$\bar{\rho} = \operatorname{argmin}_\rho \mathcal{I}(\rho).$$

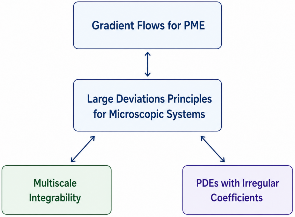
This links large deviations (determined by the particle system) with the EDI formulation of gradient flows.

Caveat, need to rewrite the rate function $\mathcal{I}$ in EDI form, that is

$$\mathcal{I}(\bar{\rho}) = \tilde{\mathcal{I}}(\bar{\rho}) = \mathcal{H}(\bar{\rho}(T)) - \mathcal{H}(\bar{\rho}(0)) + \frac{1}{2}\int_0^T |\frac{D\mathcal{H}}{D\rho}|^2_{M(\bar{\rho})} + \frac{1}{2}\int_0^T |\partial_t \bar{\rho}|^2_{M(\bar{\rho})}$$

Program to identify a gradient flow structure

- Obtain porous medium equation as hydrodynamic limit
- Prove a large deviation principle
- Prove the EDI form of the rate function

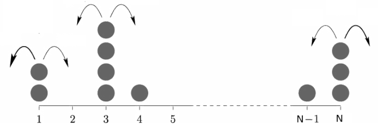


The porous medium equation as a hydrodynamic limit

Can we obtain the PME as a limit of a (stochastic) particle system?

E.g. [Suzuki, Ushiyama; 1993], [Ekhaus, Seppäläinen; 1996], [Oelschläger; 1990], [Gonçalves, Landim, Toninelli; 2009], [Gonçalves, Nahum, Simon; 2023].

The zero range process:



Local jump rate function $g(\eta) = \eta^\alpha : \mathbb{N}_0 \to \mathbb{R}_0^+$, $\alpha > 1$.

Translation invariant, asymmetric, zero mean transition probability $p(x, y) = \frac{1}{2}$.

Generator

$$L_N F(\eta) := \sum_{x,y \in \mathbb{T}_N^d} \frac{1}{2} \eta^\alpha(x)(F(\eta^{x,y}) - F(\eta)),$$

where $\eta^{x,y}$ is the configuration where one particle has moved from box x to y.

Markov jump process $\eta(t)$: $\eta(k, t) = \text{number of particles in box } k \text{ at time } t$.

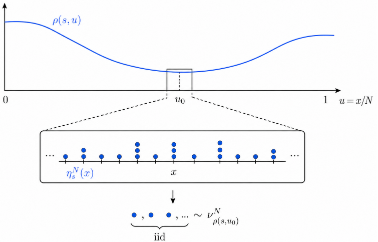
Hydrodynamic limit Empirical density field, particle sizes rescaled by χ_N,

$$\mu^N(x,t) := \chi_N\Big(\frac{1}{N}\sum_k \delta_k(x)\eta(k,t)\Big)(xN, t\frac{N^2}{\chi_N^{1-\alpha}}).$$

Martingale problem: For every $\varphi \in C^2(\mathbb{T})$,

$$\left\langle \mu_t^N, \varphi \right\rangle - \left\langle \mu_0^N, \varphi \right\rangle - \frac{1}{2}\int_0^t \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_s^N(uN)^\alpha \Delta_N\varphi(u)\ \mathrm{d}s =: M_t^{N,\chi_N}(\varphi).$$

Weak convergence $\mu^N \rightharpoonup \rho$ is not too difficult. Question: how to replace the microscopic jump rate $\eta_s(x)^\alpha$?



At macroscopic point $u = x/N$ and time s, assume *local equilibrium*

$\eta_s^N(x)$ is locally i.i.d. distributed like $\Pi^N_{\rho(s,u)}$.

Since $\mathbb{E}_{\Pi^N_{\rho(s,u)}}\eta_s^N(x)^\alpha = \rho(s,u)^\alpha$, by LLN,

$$\frac{1}{N}\sum_x \eta_s^N(x)^\alpha \Delta_N\varphi(x/N) = \frac{1}{\#\text{blocks}}\sum_{\text{blocks } B}\frac{1}{|B|}\sum_{x\in B}\eta_s^N(x)^\alpha\Delta_N\varphi(u) \approx \int_{\mathbb{T}^d}\rho^\alpha(s,u)\Delta\varphi(u)du.$$

So, as long as $\chi_N \to 0$, the hydrodynamic limit is determined by the PME

$$\partial_t\bar\rho = \Delta\bar\rho^\alpha.$$

Local equilibrium hypothesis,

$$\eta_s^N(x) \quad \text{is locally i.i.d. distributed like} \quad \nu^N_{\rho(s,u)}.$$

Rigorous implementation is the chain, in space-time averages,

$$(\eta_s(x))^\alpha \xrightarrow[\text{estimate}]{\text{one-block}} (\eta_s^\ell(x))^\alpha \xrightarrow[\text{estimate}]{\text{two-block}} (\eta_s^{\varepsilon N}(x))^\alpha \xrightarrow{\text{empirical limit}} (\rho(s,x/N))^\alpha.$$

"Replacement Lemma": For smooth φ, all $\delta > 0$, $k_\epsilon := \frac{1}{(2\epsilon)^d} 1_{[-\epsilon,\epsilon)^d}$,

$$\limsup_{\epsilon\to 0} \limsup_N \frac{\chi_N}{N^d} \log \mathbb{P}\left(\int_0^T \int_{\mathbb{T}_N^d} \Delta\varphi(t,x) \left((\eta_s^N(x))^\alpha - (\eta_s^N \star k_\epsilon(x))^\alpha \right) dxds > \delta \right) = -\infty$$

Recall:

$$\limsup_{\epsilon\to 0}\limsup_{N}\frac{\chi_N}{N^d}\log\mathbb{P}\left(\int_0^T\int_{\mathbb{T}^d_N}\Delta\varphi(t,x)\left((\eta^N_s(x))^\alpha-(\eta^N_s\star k_\epsilon(x))^\alpha\right)dxds>\delta\right)=-\infty$$

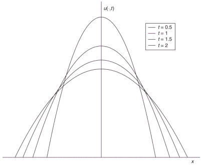
Well established for bounded, non-degenerate jump rates g (replacing $(\cdot)^\alpha$): [Kipnis-Olla-Varadhan, Benoit-Kipnis-Landim, ...]

Two difficulties:

- Superlinear growth of $g(\eta)=\eta^\alpha$, $\alpha>1$. Possible concentration of mobility $(\eta^N(x))^\alpha$.
- Degeneracy of $g(\eta)=\eta^\alpha$ at $\eta=0$. Now becomes visible with χ small. Dirichlet form degenerates.

As a result, the classical one-block, two-block approach is not applicable.

Barenblatt solution



Key difficulty: Need integrability $(\eta^N_\cdot)^\alpha\in L^{1+}_{t,x}$.

A first approach: [G., Heydecker, 2025] by a pathwise martingale estimate, with high probability,

$$\mathcal{H}(\eta^N_T)+\frac{2}{\alpha}\int_0^T\|\nabla_N(\eta^N_s)^{\alpha/2}\|^2_{L^2_x}ds\le\mathcal{H}(\eta^N_0)+\underbrace{CN^2\chi_N T}_{\text{Ito}}+\mathcal{O}_{\mathrm{LDP-}\mathbb{P}}(1).$$

Use (discrete) Sobolev embedding to conclude, so long as $N^2\chi_N\lesssim 1$.

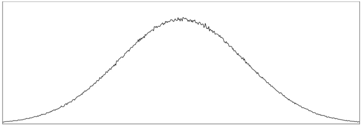


Figure 1: $N = 512$, $\chi_N = 10^{-4}$

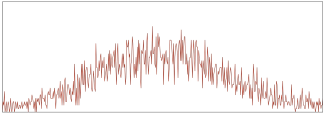
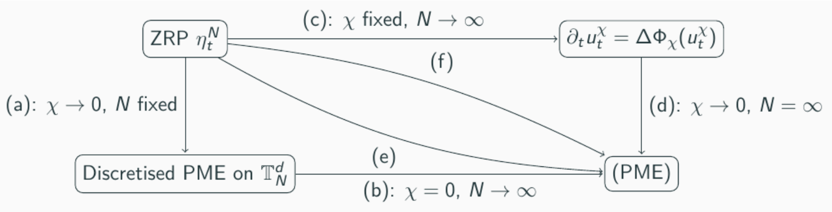


Figure 2: $N = 512$, $\chi_N = 5 \times 10^{-2}$

Fatal for pathwise regularity like $\int_0^T \|\nabla_N (\eta_s^N)^{\alpha/2}\|_{L_x^2}^2 ds \ldots$



Trajectories are regular in (e), irregular in (f).

Integrability via multi scale analysis: More noise should give more regularity on the level of laws: Dirichlet form

$$\mathfrak{D}_N(f_N) := \chi_N^{\alpha-1} \int_{X_N} \frac{1}{2} \sum_{x\sim y} (\chi_N^{-1}\eta^N(x))^\alpha \left(\sqrt{f_N(\eta^{N,x,y})} - \sqrt{f_N(\eta^N)}\right)^2 \Pi^N(d\eta^N).$$

Indeed we can show the bound [G., Heydecker; 2026],

$$\mathbb{E}_{f_N\Pi^N}\left[\left\|\nabla_N(\eta^N)^{\alpha/2}\right\|^2_{L^2_x(\mathbb{T}^d_N)}\right] \leq C\chi_N N^{2-d}\mathfrak{D}_N(f_N) + C\chi_N^\delta N^2 \mathbb{E}_{f_N\Pi^N}\left(1 + \left\|(\eta^N)^\alpha\right\|_{\ell^1(\mathbb{T}^d_N)}\right).$$

But: $\chi_N^\delta N^2$ still appears and blows up.

Idea: Local averaging $\eta^N \mapsto (\Lambda_B\eta^N)_{B\in\mathcal{P}}$, boxes B of side length ℓ_N, partitioning $\mathbb{T}^d_N$ into $\mathcal{P}$.

Two compensations:

- Averaging reduces fluctuations:

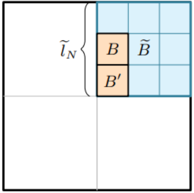
$$\text{Site-wise Variance} \sim \chi_N \mapsto \chi_N\ell_N^{-d} \sim \text{Block-wise Variance}.$$

- Smaller effective lattice size:

$$\text{True lattice size} \sim N \mapsto (N/\ell_N) \sim \text{Side length of partition}.$$

But: Only gives regularity on coarse scale ℓ_N.

Propagate regularity/integrability from large scales to small scales



Given any partition $\widetilde{B} \in \widetilde{\mathcal{P}}$, blocks length $\widetilde{\ell}_N$, refine to $\mathcal{P} \prec \widetilde{\mathcal{P}}$, blocks length $\ell_N \ll \widetilde{\ell}_N$.

Regularity within blocks:

$$\mathbb{E}_{f_N \Pi^N}\left[\left\|\nabla_{\widetilde{\ell}_N/\ell_N}(\Lambda_B \eta^N)^{\alpha/2}\right\|^2_{L^2(B\subset\widetilde{B})}\right] \leq C\widetilde{l}_N^2 N^{-d}\chi_N \mathfrak{D}_N(f_N)$$

$$+ Cl_N^{-d/2}\left(\frac{\widetilde{\ell}_N}{\ell_N}\right)^2 \chi_N^\delta \mathbb{E}_{f_N\Pi^N}\left[1 + \left\|(\eta^N)^\alpha\right\|_{\ell^1(\mathbb{T}_N^d)}\right].$$

Get control, for $p > 1$ to be chosen,

$$\|(\Lambda_B \eta^N)^\alpha\|_{L^p(B\subset\mathbb{T}_N^d)} \lesssim \|(\Lambda_{\widetilde{B}}\eta^N)^\alpha\|_{L^p(\widetilde{B}\subset\mathbb{T}_N^d)} + \ell_N^{-d/2}\chi_N^{\delta/2}\left(\frac{\widetilde{\ell}_N}{\ell_N}\right)^2 N^{d(p-1)/p} + \mathcal{O}_{\mathrm{LDP}-\mathbb{P}}(1).$$

Given any sequence $\chi_N \to 0$, we choose $p > 1$ and construct an admissable hierarchy of scales, [G., Heydecker 2026], e.g. $\chi_N = N^{-\alpha}$,

$$N = \ell_N^0 \gg \ell_N^1 \gg \cdots \gg \ell_N^K = 1.$$

Conclusion so far: The (superexponential) replacement lemma

Rate function (for simplicity neglect initial fluctuations)

$$\mathcal{I}(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}^d} |g|^2 dxds : \ g \in L^2_{t,x}, \ \underbrace{\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)}_{\text{"skeleton equation"}} \right\}$$

with domain $\rho \in C_t \mathcal{M}^1$, finite entropy dissipation $\int_0^T \int_{\mathbb{T}^d} |\nabla \rho^{\alpha/2}|^2 < \infty$.

Theorem (G., Heydecker 2026)

For every $\mathcal{O} \subseteq D([0,T], \mathcal{M}_+)$ open we have

$$e^{-\frac{N^d}{\chi_N} \inf_{\rho \in \mathcal{O}} \overline{\mathcal{I}_{|A}}(\rho)} \lesssim \mathbb{P}[\mu^N \in \mathcal{O}] \lesssim e^{-\frac{N^d}{\chi_N} \inf_{\rho \in \bar{\mathcal{O}}} \mathcal{I}(\rho)}$$

where A is the set of nice fluctuations $\mu = \rho\, dx$ with ρ a solution to

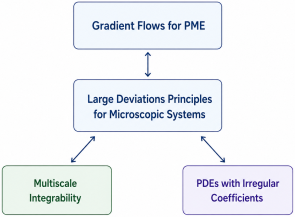
$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)$$

for some $\underline{g \in C^{1,3}_{t,x}}$. $\underline{\text{Problem:}}$ $\mathcal{I} = \overline{\mathcal{I}_{|A}}$?

This is a frequently observed problem: E.g. Fluctuations around Boltzmann equation [Rezakhanlou 1998], [Bodineau, Gallagher, Saint-Raymond, Simonella 2020].

Counter-example for Boltzmann [Heydecker; 2021], KMP process [Heydecker; 2026]

From LDP to irregular PDE



Problem:

$$\mathcal{I} = \overline{\mathcal{I}_{|A}}\,?$$

Existence of a "recovery sequence"? Given fluctuation ρ so that $\mathcal{I}(\rho dx) < \infty$, i.e. for some $g \in L^2_{t,x}$,

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

Need to find sequence of nice fluctuations $\rho^\varepsilon \in A$ so that $\rho^\varepsilon \to \rho$ and $\mathcal{I}(\rho^\varepsilon) \to \mathcal{I}(\rho)$. That is, find $g^\varepsilon \in C^{1,3}([0,T] \times \mathbb{T})$ so that $\|g^\varepsilon\|^2_{L^2} \to \|g\|^2_{L^2}$ and

$$\partial_t \rho^\varepsilon = \Delta (\rho^\varepsilon)^\alpha + \nabla \cdot ((\rho^\varepsilon)^{\frac{\alpha}{2}} g^\varepsilon)$$

satisfies $\rho^\varepsilon \to \rho$.

Skeleton equation:

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} \overbrace{g}^{\in L^2_{t,x}}) \qquad (\star)$$
$$= \nabla \cdot \rho^{\alpha/2}(2\nabla \rho^{\alpha/2} + g).$$

Scaling and criticality of the skeleton equation

Consider $(\star)$ with $g \in L^q_t L^p_x$ and $\rho_0 \in L^r_x$.

Via rescaling ("zooming in"): $p = q = 2$ is critical, and $r = 1$ is critical, $r > 1$ is supercritical.

Fehrman, G. 2023:

- Every weak solution is a kinetic solution (nonlinear commutator errors)
- Kinetic solutions are unique (exploit singular moments of the entropy-dissipation measure)

Theorem (The skeleton equation, Fehrman, G. 2023)

Let $g \in L^2_{t,x}$, ρ_0 *non-negative and* $\int \rho_0 \log(\rho_0) dx < \infty$.

1. *A function* $\rho \in L^\infty_t L^1_x$ *is a weak solution to*

$$\partial_t \rho = 2\nabla \cdot (\rho^{\frac{\alpha}{2}} \nabla \rho^{\frac{\alpha}{2}}) + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)$$

   if and only if ρ *is a renormalized entropy solution (kinetic solution).*

2. *There is a unique weak solution to*

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

   The map $g \mapsto \rho$, $L^2_{t,x} \to L^1_{t,x}$, *is weak-strong continuous. E.g. including all* $\Phi(\rho) = \rho^\alpha$, $\alpha \in [1, \infty)$.

Theorem (LDP for zero range process, G., Heydecker, 2025)

For every $\chi_N \to 0$, *the rescaled zero range process satisfies the full large deviations principle with rate function*

$$\mathcal{I}(\rho) = \inf \left\{ \|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \right\}.$$

Program to identify a gradient flow structure

- Obtain porous medium equation as hydrodynamic limit $\boxed{\checkmark}$
- Prove a large deviation principle $\boxed{\checkmark}$
- Prove the EDI form of the rate function

Gradient flow structures for the porous medium equation

Can we rewrite $\mathcal{I}$ in EDI form? That is,

$$\mathcal{I}(\rho) =_? \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\int_0^T |\frac{D\mathcal{H}}{D\rho}|^2_{M(\rho)} + \frac{1}{2}\int_0^T |\partial_t \rho|^2_{M(\rho)}$$

Informally, we have

$$\begin{aligned}
\mathcal{I}(\rho) &= \inf\left\{\int_0^T\int_{\mathbb{T}} |g|^2 dxds : \, g \in L^2_{t,x}, \, \partial_t\rho = \Delta\rho^\alpha + \nabla\cdot(\rho^{\alpha/2}g)\right\} \\
&= \int_0^T \|\partial_t\rho - \Delta\rho^\alpha\|^2_{\dot{H}^{-1}_{\rho^\alpha}} \\
&= \int_0^T \|\partial_t\rho\|^2 - 2\int_0^T (\partial_t\rho, -\nabla_{\mathcal{M}}\mathcal{H}(\rho^\alpha))_{\dot{H}^{-1}_{\rho^\alpha}} + \int_0^T \|\Delta\rho^\alpha\|^2_{\dot{H}^{-1}_{\rho^\alpha}} \\
&=_{\text{difficult!}} \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\int_0^T \|\partial_t\rho\|^2_{\dot{H}^{-1}_{\rho^\alpha}} + \frac{1}{2}\int_0^T \|\Delta\rho^\alpha\|^2_{\dot{H}^{-1}_{\rho^\alpha}}.
\end{aligned}$$

Set

$$\mathcal{A}(\rho) = \inf\{\|g\|^2_{L^2_{t,x}} : \, \partial_t\rho + \nabla\cdot(\rho^{\alpha/2}g) = 0\} = \int_0^T \|\partial_t\rho\|^2_{\dot{H}^{-1}_{\rho^\alpha}}.$$

Theorem (Entropy dissipation equality, G., Heydecker, 2023)

Let $D_\alpha(\rho) < \infty$, $\mathcal{H}(\rho_0) < \infty$, $u_0 > 0$. Then

$$\frac{1}{2}\mathcal{I}(\rho) = \mathcal{H}_{u_0}(\rho_T) - \mathcal{H}_{u_0}(\rho_0) + \frac{1}{2}\mathcal{A}(\rho) + \frac{1}{2}\int_0^T \|\rho^{\alpha/2}(s)\|^2_{\dot{H}^1} ds.$$

If ρ is a solution to the PME, we have the energy equality

$$0 = \mathcal{H}_{u_0}(\rho_T) - \mathcal{H}_{u_0}(\rho_0) + \int_0^T \|\rho^{\alpha/2}(s)\|^2_{\dot{H}^1} ds.$$

Sketch of the proof: Use reversibility and the large deviations principle.

In equilibrium, detailed balance $\implies$ $(\mathcal{T}\eta^N_\bullet)_t := \eta^N_{T-t-}$ has the same law as the original process.

Contraction principle and uniqueness of rate functions:

$$\mathcal{I}(\mathcal{T}\rho) = \mathcal{I}(\rho)$$

for all ρ. Analyse this identity without assuming any more regularity on ρ than necessary.

Recall

$$\frac{1}{2}\mathcal{I}(\rho) = \frac{1}{2}\inf\left\{\int_0^T\int_{\mathbb{T}} |g|^2 dxds :\, g \in L^2_{t,x},\, \partial_t\rho = \Delta\rho^\alpha + \nabla\cdot(\rho^{\alpha/2}g)\right\}.$$

Optimal g is uniquely characterised by membership in

$$\Lambda_\rho := \overline{\left\{\rho^{\alpha/2}\nabla\varphi : \varphi \in C^{1,2}([0,T]\times\mathbb{T}^d)\right\}}^{L^2_{t,x}}.$$

Let $\Pi[\rho]$ be orthogonal projection to this space.

For time reversal $\rho_r := \mathcal{T}\rho$ we have

$$\begin{aligned}\partial_t\rho_r &= -\Delta\rho_r^\alpha + \nabla\cdot(\rho_r^{\alpha/2}\mathcal{T}g)\\ &= \Delta\rho_r^\alpha - \nabla\cdot\rho_r^{\frac{\alpha}{2}}(2\nabla\rho_r^{\frac{\alpha}{2}} - \mathcal{T}g).\end{aligned}$$

So optimal g for ρ_r is

$$g_{\mathrm{r}} := 2\Pi[\rho]2\nabla\rho_r^{\frac{\alpha}{2}} - \mathcal{T}g.$$

Recall: $g_{\mathrm{r}} := 2\Pi[\rho_r]2\nabla\rho_r^{\frac{\alpha}{2}} - \mathcal{T}g$. We get

$$
\begin{aligned}
0 &= \mathcal{I}(\mathcal{T}\rho) - \mathcal{I}(\rho) \\
&= \alpha\mathcal{H}(\rho_T) + \frac{1}{2}\|g_r\|^2 - \alpha\mathcal{H}(\rho_0) - \frac{1}{2}\|g\|^2 \\
&= \alpha\mathcal{H}(\rho_T) - \alpha\mathcal{H}(\rho_0) + \frac{1}{2}(\|g_r\|^2 + \|\mathcal{T}g\|^2) - \|g\|^2 \\
&= \alpha\mathcal{H}(\rho_T) - \alpha\mathcal{H}(\rho_0) + \frac{1}{4}(\|g_r + \mathcal{T}g\|^2 + \|g_r - \mathcal{T}g\|^2) - \|g\|^2 \\
&= \alpha\mathcal{H}(\rho_T) - \alpha\mathcal{H}(\rho_0) + \frac{1}{4}(\|g_r + \mathcal{T}g\|^2 + 4\|\Pi[\rho_r]2\nabla\rho_r^{\frac{\alpha}{2}} - \mathcal{T}g\|^2) - \|g\|^2 \\
&= \alpha\mathcal{H}(\rho_T) - \alpha\mathcal{H}(\rho_0) + \left\|\Pi[\rho_r]\nabla\rho^{\alpha/2}\right\|_{L^2_{t,x}}^2 + frac12\mathcal{A}(\rho) - 2\mathcal{I}(\rho).
\end{aligned}
$$

A new look at properties of the skeleton equation

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\alpha/2} g)$$

- Construction of g_{r} shows how *anti*dissipative effects can arise, since

$$\partial_t \rho_r = -\Delta \rho_r^\alpha + \nabla \cdot (\rho_r^{\alpha/2} \mathcal{T} g) = \Delta \rho_r^\alpha - \nabla \cdot (\rho_r^{\frac{\alpha}{2}} g_r).$$

- Hence why L^p_x estimates had to be false: trajectories with $\rho_0 \notin L^p_x$, $\rho_T \in C^\infty_x$ give reversal $\rho_0 \in C^\infty_x$ but $\rho_T \notin L^p_x$.

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