

# Path-by-path regularization by noise for scalar conservation laws

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[G., Souganidis; CMS, 2014], [G., Souganidis; CPAM, 2016],  
[G., Perthame, Souganidis; SINUM, 2016], [Gassiat, G.; arXiv:1609.07074],  
[Maurelli, G.; arXiv:1701.05393], [Chouk, G.; arXiv:1512.06056].

# Outline

- 1 Introduction
- 2 Regularization by noise for nonlinear SPDE
- 3 Path-by-path regularization by noise
- 4 A path-by-path scaling condition

- Classical well-posedness for ODE:

$$dX_t^x = b(X_t^x)dt, \quad X_0^x = x$$

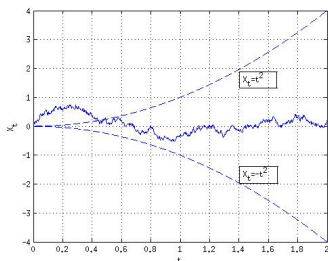
is well-posed if  $b$  is sufficiently smooth, e.g. Lipschitz continuous.

- In contrast, well-posedness for SDE: ( $\sigma > 0$ )

$$dX_t^x = b(X_t^x)dt + \sigma d\beta_t, \quad X_0^x = x$$

has a unique solution if  $b$  is bounded, measurable. This is called '*well-posedness by noise*'.

- A simple example:  $b(x) = 2\text{sgn}(x)\sqrt{|x|}$ :



# Introduction

- Key hope in SPDE: Establish similar effects for PDE, in particular in fluid dynamics, e.g. 3d-Navier-Stokes equations, gas dynamics.
- Problem: A-priori unclear which form of noise to consider
- Additive noise, e.g.

$$du = \Delta u dt + f(u) dt + dW_t \quad [\text{Gyöngy, Pardoux; 1993}]$$

$$du + (u \cdot \nabla) u dt + \nabla p dt = \Delta u dt + dW_t \quad [\text{Flandoli, Romito; 2007}].$$

- More recent: Linear multiplicative noise

$$du + b(x) \cdot \nabla u dt = \nabla u \circ d\beta_t. \quad [\text{Flandoli, Gubinelli, Priola; 2010}].$$

- We recall: Consider

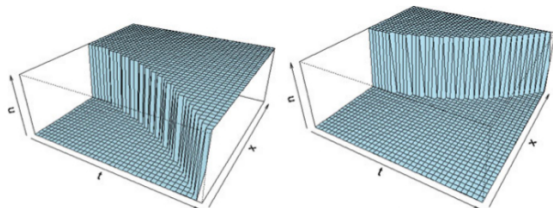
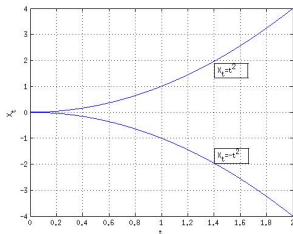
$$\partial_t u + b(x) \cdot \nabla u = 0, \quad (\text{TE})$$

for non-Lipschitz  $b$  (but, say, Hölder continuous). E.g.  $b(x) = 2\text{sgn}(x)\sqrt{|x|}$ .

- Characteristics for (TE):

$$dX_t^x = b(X_t^x)dt \in \mathbb{R}^d.$$

- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than  $u(t) \in BV$  even if  $u_0$  is smooth.
- Characteristics branch causing non-uniqueness of weak solutions.



The solution  $u^{(-)}(x, t) = 1_{\{x > -t^2\}}$

The solution  $u^{(+)}(x, t) = 1_{\{x \geq t^2\}}$

- Question: Can noise restore uniqueness or increase regularity?

- Consider,  $\sigma > 0$ ,

$$du + b(x) \cdot \nabla u = \sigma \nabla u \circ d\beta_t. \quad (\text{STE})$$

- Characteristics for (STE):

$$dX_t^x = b(X_t^x)dt - \sigma d\beta_t \in \mathbb{R}^d.$$

- Two (related) effects: regularization by noise, well-posedness by noise.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; 2010]: Weak solutions to (STE) are unique.
- Regularization by noise [Flandoli, Fedrizzi; 2013]: If  $u_0 \in \bigcap_{p \geq 1} W^{1,p}$  then  $u(t) \in \bigcap_{p \geq 1} W^{1,p}$ ,  $\mathbb{P}$ -a.s..

- Left open: What about the nonlinear case, e.g. Burgers?
- Linear multiplicative noise does not help anymore:

$$du + \partial_x u^2 dt = \partial_x u \circ d\beta_t.$$

Then:  $v(t, x) := u(t, x - \beta_t)$  is a solution to

$$\partial_t v + \partial_x v^2 = 0.$$

- In particular, weak solutions are non-unique.
- Conclusion in [Flandoli, Gubinelli, Priola; *Invent. Math.*, 2010]:  
*„It is very easy to produce examples [...] for a stochastic version of Euler equation which show that the particular noise we use does not have any regularizing effect in this case. [...] The generalization to nonlinear transport equations, where  $b$  depends on  $u$  itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem.“*
- Different forms of noise?

# Regularization by noise for nonlinear SPDE

## Regularization by noise in nonlinear SPDE



# Regularity of solutions for stochastic SCL

- Consider mean field equations

$$dX_t^i = \sigma^L \left( X_t^i, \frac{1}{L} \sum_{j=1}^L \delta_{X_t^j} \right) \circ d\beta_t \quad \text{in } \mathbb{R}^N$$

Taking  $L \rightarrow \infty$  and  $\sigma^L \rightarrow \sigma$  leads to stochastic scalar conservation laws

$$du + \operatorname{div} \underbrace{(\sigma(x, u)u \circ d\beta)}_{=: A(x, u)} = 0 \quad \text{on } (0, T) \times \mathbb{R}^d.$$

- Methods apply to general spatially homogeneous and truly nonlinear flux  $A$ .
- For simplicity, in this talk restrict to

$$du + \frac{1}{2} \partial_x u^2 \circ d\beta_t = 0.$$

Consider

$$\begin{aligned}\partial_t u + \frac{1}{2} \partial_x u^2 &= 0, \quad \text{on } (0, T) \times \mathbb{R}^d \\ u(0) &= u_0 \in L^\infty(\mathbb{R}^d).\end{aligned}$$

For

$$\chi(t, x, v) = \chi(u(t, x), v) = 1_{v < u(t, x)} - 1_{v < 0}$$

we get the kinetic form

$$\partial_t \chi + v \partial_x \chi = \partial_v m \quad \text{on } (0, T) \times \mathbb{R}^d \times \mathbb{R}.$$

Dissipation-dispersion approximations lead to

**Definition (De Lellis, Otto, Westdickenberg, 2003)**

A function  $u \in L^\infty([0, T] \times \mathbb{R}^d)$  is said to be a quasi-solution if  $\chi(t, x, v) = \chi(u(t, x), v)$  satisfies

$$\partial_t \chi + v \partial_x \chi = \partial_v m \quad \text{on } (0, T) \times \mathbb{R}^d \times \mathbb{R}$$

for some finite (signed) measure  $m$ .

Theorem (De Lellis, Westdickenberg, 2003; Jabin, Perthame 2002)

Consider

$$\partial_t u + \frac{1}{2} \partial_x u^2 = 0, \quad \text{on } (0, T) \times \mathbb{R}.$$

Then

- ① Each quasi-solution satisfies, for all  $\lambda \in (0, \frac{1}{3})$ ,

$$u \in L^1([0, T]; W^{\lambda, 1}(\mathbb{R})).$$

- ② For each  $\lambda > \frac{1}{3}$  there exists a quasi-solution  $u$ , such that  $u$  is a weak solution and

$$u \notin L^1([0, T]; W^{\lambda, 1}(\mathbb{R})).$$

Theorem (G., Souganidis; CPAM, 2016)

Let  $u \in L^\infty$  be a bounded quasi-solution to

$$du + \frac{1}{2} \partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T}.$$

Then,

$$u \in L_t^1 W_x^{\lambda,1} \quad \text{for all } \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.}$$

If  $u$  is an entropy solution, then

$$u(t) \in W_x^{\lambda,1} \quad \text{for all } t > 0, \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.} \quad (\star)$$

Two resulting questions:

- 1 Can the zero set in  $(\star)$  be chosen uniformly in  $t$ ?
- 2 Characterize the properties of Brownian paths leading to  $(\star)$ .

# Regularization by nonlinear noise

- Consider, for  $w \in C([0, T])$ ,

$$du + \frac{1}{2} \partial_x u^2 \circ dw_t = 0, \quad \text{on } \mathbb{R}.$$

- Get

$$\|u(t)\|_{W_x^{1,\infty}} \leq \left( \max_{0 \leq s \leq t} (w(s) - w(t)) \wedge (w(t) - \min_{0 \leq s \leq t} w(s)) \right)^{-1}.$$

- Decisive path property: “Changing sign of the derivative”.
- For  $w = \beta$  we get

$$v(t) \in W^{1,\infty}, \quad \mathbb{P} - a.s.$$

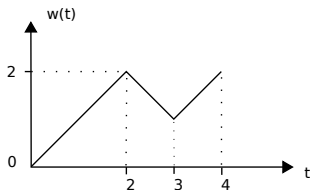
- But: Zero set depends on time  $t > 0$ .

# Regularization by nonlinear noise

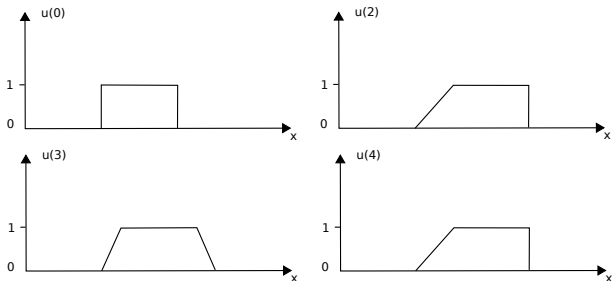
- Example:

$$\partial_t u + \frac{1}{2} \partial_x u^2 \circ dw_t = 0$$

$$u(0) = 1_{[0,1]}$$



- Solution  $u$ :



# Path-by-path regularization by noise

**Path-by-path regularization by noise**

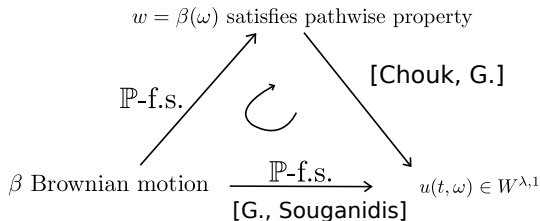
# Framework

- Model example:

$$\partial_t u + \frac{1}{2} \partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{T},$$

with  $w \in C([0, T]; \mathbb{R})$ .

- Again: Results are given for general truly nonlinear flux  $A$ .
- How to classify pathwise properties of  $w$  leading to improved regularity?





# Idea of the proof

- Ideas of the proof of regularity for

$$\partial_t u + \frac{1}{2} \partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T}.$$

- Kinetic formulation:

$$d\chi + v \partial_x \chi \circ d\beta_t = \partial_v m,$$

for some finite Radon measure  $m$ .

- Change of variables gives

$$\chi(t, x, v) = \chi_0(x + v\beta_t, v) + \int_0^t \partial_v m(s, x + v(\beta_t - \beta_s), v) ds.$$

# Idea of the proof

- Averaging over velocity

$$u(t, x) = \int_{\mathcal{V}} \chi = \int_{\mathcal{V}} \chi_0(x + v\beta_t, v) dv + \int_0^t \int_{\mathcal{V}} \partial_v m(s, x + v(\beta_t - \beta_s), v) dv ds.$$

- The averaging effect appears since the velocity average in  $v$  contains averaging of the  $x$ -variable.
- Rigorously, this can be seen by Fourier transform, that is,

$$\hat{u}(t, n) = \int_{\mathcal{V}} e^{-iv\beta_t n} \hat{\chi}_0(n, v) dv + \int_0^t \int_{\mathcal{V}} e^{-iv(\beta_t - \beta_s)n} \partial_v \hat{m}(s, n, v) dv ds.$$

- The oscillatory integrals have a regularizing effect, both in  $v$  and in  $\beta_t - \beta_s$ .

# Framework

- For SDE this has been considered by [Catellier, Gubinelli; SPA, 2016]: A path  $w \in C(\mathbb{R}_+; \mathbb{R}^d)$  is said to be  $(\rho, \gamma)$ -irregular if

$$\left| \int_s^t e^{i\langle a, w_r \rangle} dr \right| \lesssim (1 + |a|)^{-\rho} |t - s|^\gamma \quad \forall a \in \mathbb{R}^d, s < t.$$

- Note:

$$\int_s^t e^{i\langle a, w_r \rangle} dr = \int_{\mathbb{R}} e^{i\langle a, x \rangle} dL_w^{s,t}(x) = L_w^{\hat{s},t}(a)$$

the Fourier transform of the local time.

# Main result

## Theorem

Let  $w \in C^\eta([0, T], \mathbb{R}^d)$  for some  $\eta > 0$  be  $(\rho, \gamma)$ -irregular,  $u$  a bounded quasi-solution solution to

$$\partial_t u + \frac{1}{2} \partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{T}.$$

Then, for all

$$\lambda < \frac{\rho(\eta + 1) - (1 - \gamma)}{(\rho \vee 1)(\eta + 1) + (1 - \gamma)},$$

we have

$$\|u\|_{L_t^1 W_x^{\lambda, 1}} < \infty.$$

## Corollary

Let  $\beta^H$  be a fractional Brownian motion with Hurst parameter  $H \in (0, \frac{1}{2}]$  and  $u$  be a bounded quasi-solution to

$$\partial_t u + \frac{1}{2} \partial_x u^2 \circ d\beta_t^H = 0 \quad \text{on } \mathbb{T}. \quad (1)$$

Then, for all  $\lambda < \frac{1}{1+2H}$ ,

$$\|u\|_{L_t^1 W_x^{\lambda,1}} < \infty.$$

- Note: Fully recover the probabilistic result from [G., Souganidis; *CPAM*, 2016]: For  $H = \frac{1}{2}$  get  $\lambda < \frac{1}{2}$ .

# A path-by-path scaling condition

**A path-by-path scaling condition**

# Discussion of the path classification

- The proof given in [G., Souganidis; *CPAM*, 2016] uses the *scaling property* of Brownian motion and independence of increments.
- However:  $(\rho, \gamma)$ -irregularity depends on two parameters, also encoding a time regularity. Hence, does not seem to be optimal.
- Moreover:  $(\rho, \gamma)$ -irregularity not easy to check.
- To avoid the use of oscillatory integrals: Completely avoid Fourier methods in the proof.

# Idea of the proof

- Consider

$$\partial_t u + \frac{1}{2} \partial_x u^2 \circ dw_t = 0.$$

- Kinetic form

$$\partial_t \chi(t, x, v) + v \partial_x \chi(t, x, v) \circ dw_t = \partial_v m(t, x, v).$$

- Rewrite as, for  $\lambda > 0$ ,

$$\partial_t \chi(t, x, v) + v \partial_x \chi(t, x, v) \circ dw_t + \lambda \chi(t, x, v) = \partial_v m(t, x, v) + \lambda \chi(t, x, v).$$



## Idea of the proof

- Change of variables

$$\begin{aligned} \chi(t, x, \nu) &= e^{-\lambda t} \chi(0, x - \nu w_{0,t}, \nu) + \int_0^t e^{-\lambda(t-s)} (\partial_\nu m)(s, x - \nu w_{s,t}, \nu) ds \\ &\quad + \lambda \int_0^t e^{-\lambda(t-s)} \chi(s, x - \nu w_{s,t}, \nu) ds. \end{aligned}$$

- Introduce the random X-ray transform

$$(Tg)(t, x) := \int_0^t \int_\nu g(s, x - \nu w_{s,t}, \nu) e^{-\lambda(t-s)} d\nu ds$$

- Hence,

$$u := \int_\nu \chi = T(\partial_\nu m) + \lambda T\chi.$$

where  $m$  is a finite measure and  $\chi(t, x, \nu) := 1_{[0, u(t, x)]}(\nu)$ .

- Strategy: Estimate the regularity of  $T(\partial_\nu m)$ ,  $T\chi$  then use real interpolation.

# Path-by-path scaling condition

- This leads to: *Path-by-path scaling condition*: Assume that there is a  $\iota \in [\frac{1}{2}, 1]$  such that for every  $\sigma \in [0, 1)$ ,  $\lambda \geq 1$  we have

$$\int_0^T dr \int_0^{T-r} dt e^{-\lambda t} \underbrace{|w_{t+r} - w_r|}_{=: w_{r,r+t}}^{-\sigma} \lesssim \lambda^{-1+\iota\sigma}.$$

- Easy to see:  $(\rho, \gamma)$ -irregularity implies path-by-path scaling.

## Theorem

Let  $u$  be a bounded quasi-solution to

$$\partial_t u + \frac{1}{2} \partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{R}$$

and suppose that  $w \in C^\eta$  satisfies path-by-path scaling. Then, for all  $\lambda < \frac{1+\eta-\iota}{1+\eta+\iota}$ ,

$$\|u\|_{L_t^1 W_x^{\lambda,1}} < \infty.$$

# Thanks

**Thanks!**