Well-posedness by noise for PDE and symmetry breaking in cell motility

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joint work with: Paul Gassiat, Mario Maurelli
[Gassiat, G.; arxiv],[G., Maurelli.; ongoing].

Outline

Introduction

- General aim: Regularization or well-posedness by inclusion of stochastic perturbations
- Classical well-posedness for ODE:

$$dX_t^x = b(X_t^x)dt$$
$$X_0^x = x$$

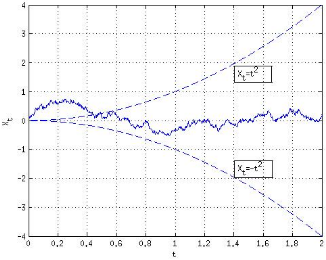
is well-posed if b is sufficiently smooth, e.g. Lipschitz continuous.

- Di Perna, Lions; Ambrosio: $b \in BV$, $(\mathrm{div} b) \in L^\infty_{loc}$.
- In contrast, well-posedness for SDE: $(\sigma > 0)$

$$dX_t^x = b(X_t^x)dt + \sigma dW_t$$
$$X_0^x = x$$

has a unique solution if b is bounded, measurable. This is called '*well-posedness by noise*'.

- A simple example: $b(x) = \operatorname{sgn}(x)\sqrt{|x|}$:



- One reason: Fokker-Planck equation for the law $u(t,x) = \mathscr{L}(X_t^x)$

$$\partial_t u = \frac{\sigma^2}{2}\Delta u + \operatorname{div}(bu).$$

Zero noise limits

- Again consider

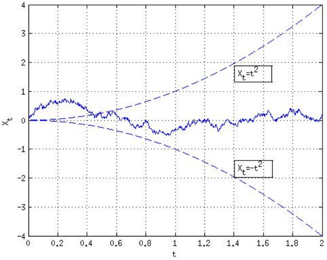
$$dX_t^x = b(X_t^x)dt + \sigma dW_t$$
$$X_0^x = x,$$

with $b(x) = \mathrm{sgn}(x)\sqrt{|x|}$.

- We may select solutions among the set of solutions to the non-perturbed problem by considering the zero noise limit $\sigma \to 0$.
- One can show e.g. [Flandoli, Delarue; 2013]

$$\mathscr{L}(X^0_\cdot) \rightharpoonup \frac{1}{2}\delta_{x^+(\cdot)} + \frac{1}{2}\delta_{x^-(\cdot)}$$

with $x^\pm$ the extremal solutions.



Introduction

- Problem: A-priori unclear which form of noise to consider
- In the case of SPDE there is more flexibility in the choice of the form of the noise
- Additive noise, e.g.

$$du = \Delta u\, dt + f(u)\, dt + dW_t \quad \text{[Gyöngy, Pardoux; 1993]}$$

$$du + (u \cdot \nabla) u\, dt + \nabla p\, dt = \Delta u\, dt + dW_t \quad \text{[Flandoli, Romito; 2007].}$$

- More recent: Linear multiplicative noise

$$du + b(x) \cdot \nabla u\, dt = \nabla u \circ d\beta_t. \quad \text{[Flandoli, Gubinelli, Priola; 2010].}$$

Introduction

- We recall: Consider
$$\partial_t u + b(x)\cdot\nabla u = 0, \tag{TE}$$
for non-Lipschitz b (but, say, Hölder continuous). E.g. $b(x) = \mathrm{sgn}(x)\sqrt{|x|}$.
- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than $u(t) \in BV$ even if u_0 is smooth.
- Characteristics branch causing non-uniqueness of weak solutions.
- Question: Can noise restore uniqueness or increase regularity?

Introduction

- Consider
$$du + b(x) \cdot \nabla u = \sigma \nabla u \circ d\beta_t, \qquad \text{(STE)}$$
with $\sigma > 0$.
- Two (related) effects: regularization by noise, well-posedness by noise.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; 2010]: Weak solutions to (STE) are unique.
- Regularization by noise [Flandoli, Fedrizzi; 2013]: If u_0 is smooth then $u(t)$ is smooth.

Introduction

- As for ODE this may be used to obtain selection principles for the deterministic case.
- Again consider

$$du + b(x) \cdot \nabla u = 0$$
$$u(0) = 1_{[0,\infty)}$$

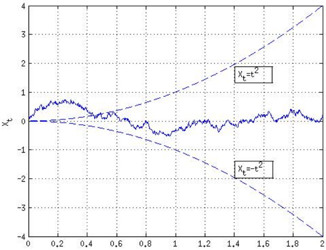
with $b(x) = \mathrm{sgn}(x)\sqrt{|x|}$.
- Then there are multiple weak solutions.
- Vanishing viscosity :

$$du^\varepsilon + b(x) \cdot \nabla u^\varepsilon = \varepsilon \Delta u^\varepsilon$$

- Then:

$$u^\varepsilon \to u = \frac{1}{2} u_1 + \frac{1}{2} u_2$$
$$= 1_{\{x \geq x^+\}} + \frac{1}{2} 1_{\{x^- < x < x^+\}},$$

with $u_1 = 1_{\{x \geq x^+\}}$, $u_2 = 1_{\{x > x^-\}}$.



Introduction

- Zero noise limit [Attanasio, Flandoli; 2009]:

$$du^\sigma + b(x)\cdot\nabla u^\sigma = -\sigma\nabla u^\sigma \circ dW_t.$$

Then:

$$\mathscr{L}(u^\sigma) \to \frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2}$$

with $u_1 = 1_{\{x\geq x^+\}}$, $u_2 = 1_{\{x>x^-\}}$.

Introduction

- Left open: What about the nonlinear case, e.g. Burgers?
- Linear multiplicative noise does not help anymore:

$$du + \partial_x u^2 \, dt = \partial_x u \circ d\beta_t.$$

- Then: $v(t,x) := u(t, x - \beta_t)$ is a solution to

$$\partial_t v + \partial_x v^2 = 0.$$

- In particular, weak solutions are non-unique.

Well-posedness by noise for stochastic scalar conservation laws

Well-posedness by noise for stochastic scalar conservation laws

Introduction

Consider

$$\partial_t u + b(x) \cdot \nabla(u^2) = 0,$$

for irregular b (in particular $\mathrm{div} b \notin L^\infty$):

- The deterministic problem is ill-posed in general (entropy solutions are non-unique)
- Can we restore well-posedness by adding a linear multiplicative noise term?
- Non-trivial: shocks due to the nonlinearity and shocks due to the irregularity of b may combine in such a way that this noise may be insufficient.

Stochastic Burgers' equation:

$$du + b(x) \cdot \nabla(u^2) dt = \nabla u \circ d\beta_t.$$

Theorem

Assume $b \in (W^{1,1} \cap L^\infty)(\mathbb{R}^d)$ and $\operatorname{div} b \in L^p(\mathbb{R}^d)$ for some $p > d$. Then the stochastic Burgers' equation admits a unique entropy solution.

Model example: $b(x) = \operatorname{sgn}(x)(|x|^{1/2} \wedge K)$, some $K > 0$.

Idea of the proof: Given an entropy solution u introduce a new (velocity) variable $\xi \in \mathbb{R}$ and define the kinetic function:

$$f = f[u](t,\omega,x,\xi) = 1_{0<u(t,\omega,x)<\xi}.$$

Then informally

$$\partial_t f + 2b\xi \cdot \nabla_x f + \nabla_x f \circ d\beta_t = \partial_\xi m$$

for some nonnegative random measure m on $[0,T] \times \mathbb{R}^d_x \times \mathbb{R}_\xi$. Kinetic equation in Itô form:

$$\partial_t f + 2b\xi \cdot \nabla_x f + \nabla_x f \, d\beta_t - \frac{1}{2}\Delta_x f = \partial_\xi m$$

A Laplacian appears, which suggests regularization. Note: The equation is hyperbolic (not parabolic: no regularization of initial datum). Trick: First renormalize, need $b \in W^{1,1}$ for commutator estimates [DiPerna-Lions 89, Ambrosio 04].

Zero noise limits

- Aim: What happens in zero noise limit for

$$du^\sigma + b(x) \cdot \nabla (u^\sigma)^2 = \sigma \nabla u^\sigma \circ d\beta_t?$$

Which of the possibly non-unique solutions is selected?

Regularization by noise for stochastic Hamilton-Jacobi equations

Regularization by noise for stochastic Hamilton-Jacobi equations

Introduction

- Can we use nonlinear noise to regularize nonlinear PDE?
- Model example: Porous medium equation

$$\partial_t w = \frac{1}{6}\partial_{xx} w^3, \quad \text{on } \mathbb{R},$$

with smooth, compactly supported initial condition.

- Solutions are known to develop singularities:

$$\|\partial_x w(t)\|_{L^\infty} = \infty,$$

for all $t > 0$ large enough.

- Linear multiplicative noise does not help:

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \partial_x v \circ d\beta_t, \quad \text{on } \mathbb{R}.$$

Then $w(t,x) = v(t, x - \sigma\beta_t)$.

Introduction

- Instead, consider, for $\sigma > 0$,

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \frac{1}{2}\partial_x v^2 \circ d\beta_t, \quad \text{on } \mathbb{R}. \tag{SPME}$$

- Note: If u is the viscosity solution to

$$du = \frac{1}{6}\partial_x(\partial_x u)^3\, dt + \sigma \frac{1}{2}(\partial_x u)^2 \circ d\beta_t, \quad \text{on } \mathbb{R},$$

then, $v = \partial_x u$ solves (SPME).

Setup

- General framework: Consider

$$du = F(t,x,u,Du,D^2u)\,dt + \frac{1}{2}|Du|^2 \circ d\beta_t, \quad \text{on } \mathbb{R}^N,$$

  and compare to the regularity of the non-perturbed problem

$$dw = F(t,x,w,Dw,D^2w), \quad \text{on } \mathbb{R}^N.$$

- F satisfies the usual assumptions from the theory of stochastic viscosity solutions

Key result

- Control on the rate of loss of semiconcavity: There is a $V_F \in Lip_{loc}(\mathbb{R}_+ \setminus \{0\})$ such that, for $\ell_0 > 0$,

$$D^2 g \leq \frac{Id}{\ell_0} \;\Rightarrow\; D^2(S_F(t,g)) \leq \frac{Id}{\ell(t)},$$

where $t \mapsto S_F(t,g)$ denotes the solution to

$$\begin{aligned} \partial_t w + F(t,x,w,Dw,D^2 w) &= 0 \\ w(0) &= g \end{aligned}$$

and ℓ the solution to

$$\begin{aligned} \dot{\ell}(t) &= V_F(\ell(t)) \\ \ell(0) &= \ell_0. \end{aligned}$$

Key result

Theorem

Let u be the solution to

$$\partial_t u = F(t,x,u,Du,D^2u) + \tfrac{1}{2}|Du|^2 \circ d\beta_t,$$

with

$$D^2 u_0 \leq \frac{Id}{\ell_0} \quad \text{some } \ell_0 \in [0,\infty).$$

Then for each $t \geq 0$*, one has*

$$D^2 u(t,\cdot) \leq \frac{Id}{L_t}, \tag{1}$$

where L is the maximal solution to

$$\begin{aligned} dL_t &= V_F(L_t)dt + d\beta_t, \ \text{ with } L_t \geq 0, \\ L_0 &= \ell_0. \end{aligned} \tag{2}$$

Model example

- Return to the model example

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \frac{1}{2}\partial_x v^2 \circ d\beta_t, \quad \text{on } \mathbb{R}.$$

- Deterministic case: $\|\partial_x w(t)\|_\infty = \infty$ for all $t > 0$ large enough.
- We have the *sharp* bound

$$\|\partial_x v(t)\|_\infty \leq \frac{1}{L^+(t) \wedge L^-(t)},$$

where $L^\pm$ solve

$$dL^\pm = -\frac{1}{L^\pm(t)}\, dt \pm \sigma\, d\beta_t, \ \text{with } L^\pm_t \geq 0,$$
$$L^\pm(0) = \frac{1}{\|(\partial_x v_0)_\pm\|_\infty}.$$

Model example

- In conclusion,
  - If $\sigma^2 > 2$: For all $t \geq 0$, $\mathbb{P}$-a.s.

$$v(t) \in W^{1,\infty}$$

  - If $\sigma^2 \leq 2$: $\mathbb{P}$-a.s. for all $t > 0$ large enough

$$v(t) \notin W^{1,\infty}$$

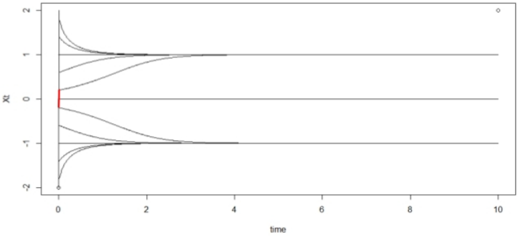
Synchronization by noise

Synchronization by noise

Finite dimensional case

- Inclusion of noise may simplify long-time behavior
- Simple example: Random motion in a double-well potential

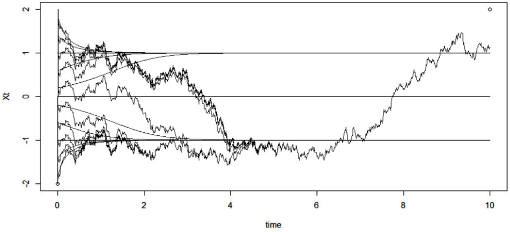
$$dX_t = (X_t - X_t^3)dt$$



Finite dimensional case

- Inclusion of noise may simplify long-time behavior
- Simple example: Random motion in a double-well potential

$$dX_t = (X_t - X_t^3)dt + \sigma dW_t$$



- Get:

$$\lim_{t\to\infty} |X_t^x - X_t^y| = 0$$

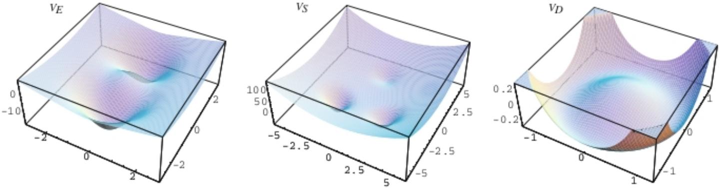
in probability.

- In $1-d$ can use order-preservation: If $x \leq y$ then

$$X_t^x \leq X_t^y \quad \forall t \geq 0.$$

Finite dimensional case

- Extension to potentials with finitely many sinks: [Martinelli, Scoppola; *CMP*, 1998], [Tearne; *PTRF*, 2008].



- [Flandoli, G., Scheutzow; 2015]: Synchronization by noise for gradient systems
$$dX_t = -\nabla V(X_t)dt + \sigma dW_t.$$
- If $\sigma > 0$, $e^{-\frac{\sigma^2}{2}V} \in L^1$, $\lambda_{top} < 0$. From this deduce that synchronization holds. Proof: Via local stability analysis.

Finite dimensional case

- How about synchronization by noise for PDE? E.g.

$$du = (\Delta u + u - u^3)dt + dW_t. \tag{RDE}$$

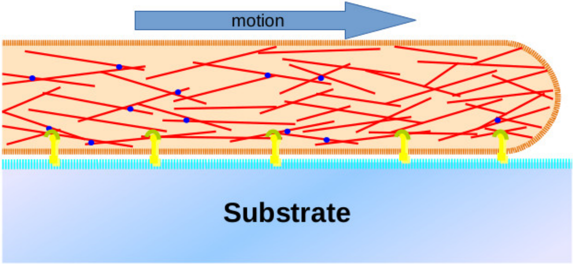
- Using comparison principles for (RDE) synchronization by noise can be shown.
- No results on synchronization by noise for systems of PDE are known.
- In contrast: Mixing behavior on the level of laws is well-understood [Hairer, Mattingly; *Ann. of Math.*, 2006; *AoP*, 2008].

SPDE in cell motility

SPDE in cell motility

SPDE in cell motility

- Reference: [Barnhart, Lee, Allen, Theriot, Mogilner; PNAS, 2015], "Balance between cell-substrate adhesion and myosin contraction determines the frequency of motility initiation in fish keratocytes."
- Spontaneous change of state: resting to moving, symmetric to non-symmetric
- Key explanation: Random fluctuations combined with nonlinear feedback between actin, myosin and adhesion.



(source: Jan Furhmann)

SPDE in cell motility

- Model: Velocity of F-actin flow U, myosin density M, effective adhesion drag coefficient ζ (effectively describes the strength of the coupling between the actin network and the substrate).
- The adhesion resistive force is balanced by the active contractile:

$$\zeta U \approx \nabla M.$$

- Passive stresses due to deformation: Viscosity

$$\nabla(\mathrm{div} U) + \Delta U + \nabla M = \zeta U$$

inside the cell $\mathscr{O}(t)$ (free boundary).

- Adhesion strength

$$\partial_t \zeta = D_\zeta \Delta \zeta - \mathrm{div}\left(\xi(t,x)\sqrt{\zeta}\right) + f(\zeta) + s(|U|) \quad \text{in } \mathscr{O}(t),$$

where $\xi(t,x)$ is a suitable random perturbation.

SPDE in cell motility

- Myosin density evolution

$$\partial_t M = D_M \Delta M - \operatorname{div}\left(U + \xi(t,x)\sqrt{M}\right) \quad \text{in } \mathscr{O}(t).$$

- Stick-slip motion: actin flow vs. adhesion strength. If actin flow becomes large, then adhesion strength caused by binding of adhesion molecules to both the actin network and the underlying substrate becomes small (inducing slip motion)
- Phenomenologically model (f,s) to reproduce this effect: Bistability of (f,s) corresponds to bistability of resting/moving cell
- Solution to $f(\zeta)+s(u)=0$:

$$\zeta = \zeta_1 \gg 1 \text{ if } u \ll u^*$$
$$\zeta = \zeta_0 \ll 1 \text{ if } u \gg u^*.$$

- Noise: Induces spontaneous change of state

SPDE in cell motility

- Difficulties and aims:
  - free boundary problem
  - low regularity (at least spatial) due to stochastic perturbation
  - stability analysis: meta-stability of resting vs. moving state

Thanks

Thanks!