

# Optimal regularity for the porous medium equation

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[G.; JEMS 2019+], [G., Sauer, Tadmor; arxiv, 2019].

# Outline

- 1 Scaling arguments and special solutions
- 2 Existing regularity results
- 3 Optimal regularity for the porous medium equation
- 4 Space-time optimal regularity for the porous medium equation

- We consider the porous medium equation

$$\begin{aligned}\partial_t u &= \Delta (|u|^{m-1} u) \text{ on } (0, T) \times \mathbb{R}^d \\ u(0) &= u_0 \text{ on } \mathbb{R}^d,\end{aligned}$$

with  $u_0 \in L^1(\mathbb{R}^d)$ ,  $m > 1$ .

- Degenerate parabolic Anderson model

$$\partial_t u = \Delta (|u|^{m-1} u) + u \xi \quad \text{on } (0, T) \times \mathbb{R}$$

with  $u_0 \in L^1(\mathbb{R})$ ,  $m \in (1, 2)$ ,  $\xi$  spatial white noise.

- Aim: Optimal regularity of solutions in (fractional) Sobolev spaces.

## Application: Cell dynamics

- Spreading of cell populations

$$\partial_t u = \operatorname{div}(\kappa \nabla u) + f(u),$$

where  $u$  is the density of cells,  $f(u)$  is the reproduction/death rate.

- If cells avoid crowding  $\kappa$  is an increasing function of the cell density,  $\kappa = \varphi(u)$  with  $\varphi$  increasing.
- In particular cases we have  $\varphi(u) = au^\gamma$ . Hence,

$$\partial_t u = \frac{a}{\gamma+1} \Delta u^{\gamma+1} + f(u),$$

- Random environment leads to the degenerate parabolic Anderson model

$$\partial_t u = \frac{a}{\gamma+1} \Delta u^{\gamma+1} + u\xi,$$

where  $\xi$  is spatial white noise.

## Application: Interacting cells

- Interacting cell system

$$\frac{d}{dt} X_t^i = -\frac{1}{L} \sum_{j=1, j \neq i}^L \nabla V_L(X_t^i - X_t^j) \quad i = 1 \dots L,$$

where  $V_L$  is a rescaled interaction potential (repelling)

$$V_L(x) = \lambda^d V_1(\lambda x), \quad \lambda = L^{\frac{\beta}{d}}$$

and  $\beta \in (0, 1)$ .

- Consider the empirical process

$$t \mapsto \mu_t^L = \frac{1}{L} \sum_{i=1}^L \delta_{X_t^i}.$$

Under regularity, decay and symmetry assumptions on  $V_1$  obtain

**Theorem (Oelschläger)**

*If  $\mu_0^L \rightharpoonup m_0(x)dx$ , then  $\mu_t^L \rightharpoonup m_t(x)dx$  and with  $c = \frac{1}{2} \int V_1(x) dx$ ,*

$$\partial_t m = c \Delta m^2, \quad m(0) = m_0.$$

See also: Lions-Mas Gallic 2001, Figalli-Philipowski 2008, Flandoli 2016, Carrillo-Craig-Papacchini 2018

# Scaling arguments and special solutions

## Scaling arguments and special solutions

- Note

$$\begin{aligned}\partial_t u &= \Delta u^{[m]} = m \operatorname{div}(|u|^{m-1} \nabla u) \\ &= m|u|^{m-1} \Delta u + m(m-1)u^{[m-2]} |\nabla u|^2.\end{aligned}$$

- Barenblatt solution:

$$U(x, t) = t^{-\alpha} F(xt^{-\beta}) = t^{-\alpha} (C - k|xt^{-\alpha/d}|^2)_+^{\frac{1}{m-1}},$$

where  $\alpha = \frac{d}{d(m-1)+2}$ ,  $k = \frac{(m-1)\alpha}{2md}$ . We observe that

$$\lim_{t \downarrow 0} U(x, t) = M\delta_0(x)$$

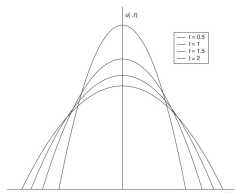


Figure: Fundamental solution of the porous medium equation

## Lemma

Assume that for some  $s \geq 0$ ,  $p \geq 1$ ,  $C \geq 0$  we have

$$\|u\|_{L^p([0, T]; \dot{W}^{s, p}(\mathbb{R}_x^d))}^p \leq C \|u_0\|_{L^1(\mathbb{R}_x^d)},$$

for all solutions  $u$  to PME. Then, necessarily  $p \leq m$  and  $s \leq \frac{2}{m}$ .

Use scale invariances:

$$\tilde{u}(t, x) := u(\eta t, x) \eta^{\frac{1}{m-1}}, \quad \tilde{u}(t, x) := u(t, \eta x) \eta^{-\frac{2}{m-1}}.$$



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## Example

Consider the Barenblatt solution

$$U(t, x) = t^{-\alpha} (C - k|x t^{-\beta}|^2)_+^{\frac{1}{m-1}}.$$

Then

$$U \in L^m([0, T]; \dot{W}^{s,m}(\mathbb{R}_x^d))$$

implies  $s < \frac{2}{m}$ .

Use  $U(t, x) = t^{-\alpha} F(x t^{-\beta})$ .

# Existing regularity results

## Existing regularity results

- **Continuity**  
Caffarelli-Friedman 1979, Sacks 1983, Caffarelli-Evans 1983, DiBenedetto 1983, Ziemer 1982
- **Hölder continuity:  $\alpha$ -Hölder continuity with  $\alpha = \frac{1}{m} \in (0, 1)$ .**  
Caffarelli-Friedman 1980, DiBenedetto-Friedman 1985, Bögelein, Duzaar, Gianazza 2014
- **Regularity of the open interface**  
Caffarelli-Friedman 1980, Caffarelli-Vazquez-Wolansky 1987, Caffarelli-Wolanski 1990, Daskalopoulos-Hamilton 1998, Koch 1999
- **Eventual  $C^\infty$  regularity**  
Aronson-Vázquez 1987, Kienzler-Koch-Vazquez 2016
- **Regularity of the pressure or powers of the solution**  
Koch 1999, Gianazza-Schwarzacher 2016
- **Time regularity (vanishing force)**  
Aronson-Bénilan 1979, Crandall-Pazy-Tartar 1979, Bénilan-Crandall 1981, Crandall-Pierre 1982
- **Regularity in Sobolev spaces**  
Lions-Perthame-Tadmor 1994, Ebmeyer 2005, Tadmor-Tao 2007

Let  $\dot{\mathcal{N}}^{s,p}$  be the homogeneous Nikolskii space ( $\dot{\mathcal{N}}^{s,p} = \dot{B}_{p,\infty}^s$ ).

Theorem (Tadmor, Tao; CPAM 2007, Ebmeyer; JMAA 2005)

Let  $u_0 \in L^2(\mathbb{R}^d)$ . Then

$$\|u\|_{L^{m+1}([0,T]; \dot{\mathcal{N}}^{\frac{2}{m+1}, m+1}(\mathbb{R}^d))}^{m+1} \leq C_m \|u_0\|_{L_x^2}^2.$$

- Note:  $\frac{2}{m+1} \leq 1$ , which is inconsistent with the linear case ( $m = 1$ ) and with the optimal regularity of the Barenblatt solution.

Consider

$$\partial_t u = \Delta u^{[m]}.$$

By (soft) energy methods may be improved to:

Theorem (G., JEMS, 2019+)

Let  $\varepsilon > 0$ ,  $m \geq 2$  and  $u_0 \in L^{1+\varepsilon}(\mathbb{R}_x^d)$ . Then

$$\|u\|_{L^{m+\varepsilon}([0, T]; \mathcal{N}^{\frac{2}{m+\varepsilon}, m+\varepsilon}(\mathbb{R}_x^d))}^{m+\varepsilon} \leq C_{\varepsilon, m} \|u_0\|_{L_x^{1+\varepsilon}}^{1+\varepsilon}.$$

- Note: optimal regularity for the Barenblatt solution, but  $m \geq 2$  implies  $\frac{2}{m+\varepsilon} < 1$ .
- Problem: How to get to more than one derivative?

# Optimal regularity for the porous medium equation

## Optimal regularity for the porous medium equation

Consider

$$\partial_t u = \frac{1}{m} \Delta u^{[m]} + S(t, x) \quad \text{on } (0, T) \times \mathbb{R}^d \quad (\text{PME})$$

with  $u_0 \in L^1(\mathbb{R}_x^d)$ ,  $S \in L^1([0, T] \times \mathbb{R}_x^d)$ .

Consider

$$\partial_t u = \frac{1}{m} \Delta u^{[m]} + S(t, x) \quad \text{on } (0, T) \times \mathbb{R}_x^d \quad (\text{PME})$$

with  $u_0 \in L^1(\mathbb{R}_x^d)$ ,  $S \in L^1([0, T] \times \mathbb{R}_x^d)$ .

Theorem (G., JEMS, 2019+)

Let  $\varepsilon > 0$ ,  $u_0 \in L^{1+\varepsilon}(\mathbb{R}_x^d)$ ,  $S \in L^{1+\varepsilon}([0, T] \times \mathbb{R}_x^d)$ . Let  $u$  be the unique entropy solution to the PME. Then, for all

$$s \in [0, \frac{2}{m}), \quad p \in [1, m)$$

we have

$$u \in L^p([0, T]; \dot{W}_{loc}^{s,p}(\mathbb{R}_x^d)).$$

In addition, for all  $\mathcal{O} \subset\subset \mathbb{R}^d$  there is a constant  $C = C(m, p, s, T, \mathcal{O})$  such that

$$\|u\|_{L^p([0, T]; \dot{W}^{s,p}(\mathcal{O}))} \leq C \left( \|u_0\|_{L_x^1}^2 + \|S\|_{L_{t,x}^1} + 1 \right).$$



**“Proof”**: A real analysis attempt

- Kinetic form [Lions, Perthame, Tadmor 1994], [Chen, Perthame; 2003]:  
Introduce

$$\chi(u(t, x), v) = 1_{v < u(t, x)} - 1_{v < 0}.$$

Then,

$$\partial_t \chi = |v|^{m-1} \Delta_x \chi + \partial_v q \text{ on } (0, T) \times \mathbb{R}_x^d \times \mathbb{R}_v,$$

for some  $q \in \mathcal{M}^+$ .

- Variation of constants/Duhamel

$$\chi(t, x, v) = e^{-|v|^{m-1} t \Delta} \chi_0(x, v) + \int_0^t e^{-|v|^{m-1} (t-r) \Delta} \partial_v q(r, x, v) dr.$$

- Decompose  $u$  in degenerate and non-degenerate part:

$$u(t, x) = \int_v \chi(u(t, x), v) = \underbrace{\int_{|v| \leq \lambda} \chi(u(t, x), v)}_{u^0(t, x)} + \underbrace{\int_{|v| \geq \lambda} \chi(u(t, x), v)}_{u^1(t, x)}.$$

- Note:

$$u^1(t, x) = \int_{|v| \geq \lambda} e^{-|v|^{m-1} t \Delta} \chi_0(x, v) + \int_0^t \int_{|v| \geq \lambda} e^{-|v|^{m-1} \Delta (t-r)} \partial_v q(r, x, v).$$

- Trivial estimate: For all  $r \geq 1$ ,

$$\|u^0\|_{L_{t,x}^r} = \left\| \int_{|v| \leq \lambda} \chi(u(t,x), v) \right\|_{L_{t,x}^r} \lesssim \lambda.$$

- Recall:

$$u^1(t,x) = \int_{|v| \geq \lambda} e^{-|v|^{m-1}t\Delta} \chi_0(x,v) + \int_0^t \int_{|v| \geq \lambda} e^{-|v|^{m-1}\Delta(t-r)} \partial_v q(r,x,v).$$

- Heat kernel estimates: For  $\alpha < 1$ ,

$$\|u^1\|_{L_t^1 H_x^{2\alpha,1}} \lesssim \lambda^{-1-\alpha(m-1)} \|q\|_{\mathcal{M}_{t,x,v}}.$$

- Test case:  $m = 1$ ,  $\alpha = 1$ , get  $u \in L_t^1 W_x^{1,1}$ .

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- Test case:  $m = 1$ ,  $\alpha = 1$ , get  $u \in L_t^1 W_x^{1,1}$ .
- Singular moments

$$\|u^1\|_{L_t^1 H_x^{2\alpha,1}} \lesssim \lambda^{-1+\gamma-\alpha(m-1)} \| |v|^{-\gamma} q \|_{\mathcal{M}_{t,x,v}}.$$

- For  $r = 1$ ,  $\alpha = 1 - \varepsilon$ : Gives ( $u = u^0 + u^1$ )

$$u \in (L_t^1 L_x^1, L_t^1 H_x^{2\alpha,1})_{\theta, \infty} \subseteq L_t^1 W_x^{\frac{2}{m}-\varepsilon, 1}.$$

- Recall: For all  $r \geq 1$ ,

$$\|u^0\|_{L_{t,x}^r} = \left\| \int_{|v| \leq \lambda} \chi(u(t,x), v) \right\|_{L_{t,x}^r} \lesssim \lambda$$

$$\|u^1\|_{L_t^1 H_x^{2\alpha,1}} \lesssim \lambda^{-1+\gamma-\alpha(m-1)} \| |v|^{-\gamma} q \|_{\mathcal{M}_{t,x,v}}.$$

- Real interpolation: Problem  $L_t^1 H_x^{2\alpha,1} \hookrightarrow L_t^r L_x^r$  only if  $r = 1$ , otherwise  $(L_t^r L_x^r, L_t^1 H_x^{2\alpha,1})_{\theta, \infty}$  not controlled.
- No optimal integrability.

- Idea: Micro-local decomposition of the Fourier-space depending on the degeneracy in  $|v|^{m-1}$ .
- Aim: Micro-local decomposition is chosen so that all regularity is on  $\tilde{u}^0$ , while  $\tilde{u}^1$  is only  $L^1_{t,x}$ :

$$u \in \underbrace{(L^r_t H^{2\alpha,r}_x)}_{\ni \tilde{u}^0}, \underbrace{L^1_{t,x}}_{\ni \tilde{u}^1} \theta_{,\infty} \subseteq L^{m-\varepsilon}_t W^{\frac{2}{m}-\varepsilon, m-\varepsilon}_x.$$

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- Aim: Micro-local decomposition is chosen so that all regularity is on  $\tilde{u}^0$ , while  $\tilde{u}^1$  is only  $L^1_{t,x}$ :

$$u \in \underbrace{(L^r_t H^{2\alpha,r}_x)}_{\ni \tilde{u}^0}, \underbrace{L^1_{t,x}}_{\ni \tilde{u}^1} \theta_{\infty} \subseteq L^{m-\varepsilon}_t W^{2/m-\varepsilon, m-\varepsilon}_x.$$

- Recall: Kinetic form for  $\chi(u(t,x), v) = 1_{v < u(t,x)} - 1_{v < 0}$

$$\partial_t \chi = |v|^{m-1} \Delta \chi + \partial_v q \text{ on } (0, T) \times \mathbb{R}^d_x \times \mathbb{R}_v,$$

for some  $q \in \mathcal{M}^+$ .

- Fourier transformation in time and space (modulo cut-off in time)

$$\underbrace{i\tau \hat{\chi} - |v|^{m-1} |\xi|^2 \hat{\chi}}_{=: \mathcal{L}(i\tau, \xi, v) \hat{\chi}} = \partial_v \hat{q}.$$

- Hence, informally,

$$\hat{\chi} = \frac{1}{i\tau - |v|^{m-1} |\xi|^2} \partial_v \hat{q} = \frac{1}{\mathcal{L}(i\tau, \xi, v)} \partial_v \hat{q}.$$

- Gain regularity, depending on the degeneracy of the operator  $\mathcal{L}(i\tau, \xi, v)$ .

- Micro-local decomposition:

$$\phi_0(\xi) + \sum_{j \geq 1} \phi_1(2^{-j}\xi) = 1.$$

Decompose  $\chi$  by

$$\hat{\chi} = \underbrace{\phi_0\left(\frac{\mathcal{L}(i\tau, \xi, \nu)}{\delta}\right)}_{\chi^0} \hat{\chi} + \sum_{j \geq 1} \underbrace{\phi_1\left(\frac{\mathcal{L}(i\tau, \xi, \nu)}{2^j \delta}\right)}_{\chi_j^1} \hat{\chi}.$$

- Paley-Littlewood decomposition (in space) to work on fixed blocks of Fourier modes.
- On non-degenerate parts use the equation ( $\hat{\chi} = \frac{1}{\mathcal{L}(i\tau, \xi, \nu)} \partial_\nu \hat{q}$ ) and velocity-average.
- Establish multiplier estimates to control regularity of  $\chi^0$ .

## Obstacles:

- 1 Integrability: Established methods yield good estimates only in an  $L^2$ -framework. This prevents from obtaining optimal integrability exponents  
 -> Introduce a new notion of isentropic truncation properties for Fourier multipliers.
- 2 Established methods can only make use of the fact that  $q$  has finite mass. This necessarily leads to sub-optimal estimates.  
 -> Solution: Use that  $q$  allows singular moments  $\int |v|^{-1+} dq < \infty$ .
- 3 Bootstrapping: Established methods rely on bootstrapping, i.e. assuming that  $u \in W_x^{\alpha,1}$  for some  $\alpha$  use that  $\chi(u) \in W_{x,v}^{\alpha,1}$ . But: This is true for  $\alpha \leq 1$  only!



## Space-time optimal regularity for the porous medium equation

What was left open so far:

- Space-*time* regularity
- Initial data in  $L^1(\mathbb{R}_x^d)$   $\rightarrow$  application to the Barenblatt solution
- Higher order integrability & non-homogeneous estimates

## Theorem (G., Sauer, Tadmor; 2019)

Let  $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ ,  $S \in L^1([0, T] \times \mathbb{R}^d) \cap L^p([0, T] \times \mathbb{R}^d)$  for some  $p \in [1, \infty)$  and assume  $m \in (1, \infty)$ . Let  $u$  be the unique entropy solution. Let  $\rho \in (\rho, m-1+\rho)$  and define

$$\kappa_t := \frac{m-1+\rho-p}{\rho} \frac{1}{m-1}, \quad \kappa_x := \frac{p-\rho}{\rho} \frac{2}{m-1}.$$

Then

- 1 For all  $\sigma_t \in [0, \kappa_t)$  and  $\sigma_x \in [0, \kappa_x)$  we have

$$u \in W^{\sigma_t, p}(0, T; W^{\sigma_x, p}(\mathbb{R}^d)).$$

- 2 Let  $s \in [0, 1]$  and define

$$p := s(m-1) + 1, \quad \kappa_t := \frac{1-s}{s(m-1)+1}, \quad \kappa_x := \frac{2s}{s(m-1)+1}.$$

Then for all  $\sigma_t \in [0, \kappa_t)$ ,  $\sigma_x \in [0, \kappa_x)$  and  $q \in [1, p]$  we have

$$u \in W^{\sigma_t, q}(0, T; W^{\sigma_x, q}(\mathcal{O})).$$

## Difficulties:

- Identifying the right anisotropic fractional spaces
  - Fourier analytic method works nicely for *homogeneous* Besov spaces only
  - Leads to Schmeisser, Triebel's dominating mixed anisotropic Besov spaces
  - Embedding to non-homogeneous, standard Sobolev spaces delicate
- $L^1$ -data: Singular moments  $\int |v|^{-\gamma} q$ ,  $\gamma \in (0, 1)$  not finite anymore
  - Respect the different origin of difficulty at the degeneracy  $|v| = 0$  and the singularity at  $|v| = \infty$ .

## Identifying the right spaces

### Definition

Let  $\sigma_i \in (-\infty, \infty)$ ,  $i = t, x$ ,

- 1 The homogeneous Besov space with dominating mixed derivatives  $S_{p,\infty}^{\bar{\sigma}} \dot{B}(\mathbb{R}^{d+1})$  is given by

$$S_{p,\infty}^{\bar{\sigma}} \dot{B} := S_{p,\infty}^{\bar{\sigma}} \dot{B}(\mathbb{R}^{d+1}) := \{f \in \mathcal{L}^p : \|f\|_{S_{p,\infty}^{\bar{\sigma}} \dot{B}} < \infty\},$$

with the norm

$$\|f\|_{S_{p,\infty}^{\bar{\sigma}} \dot{B}} := \sup_{l,j \in \mathbb{Z}} 2^{\sigma_t l} 2^{\sigma_x j} \|\mathcal{F}_{t,x}^{-1} \eta_l \varphi_j \mathcal{F}_{t,x} f\|_{L^p(\mathbb{R}^{d+1})}.$$

### Lemma

Let  $\sigma_t, \sigma_x > 0$  and  $p \in [1, \infty]$ . Then

$$\left( L^p(\mathbb{R}^{d+1}) \cap \tilde{L}_x^p \dot{B}_{p,\infty}^{\sigma_t} \cap \tilde{L}_t^p \dot{B}_{p,\infty}^{\sigma_x} \cap S_{p,\infty}^{\bar{\sigma}} \dot{B} \right) = S_{p,\infty}^{\bar{\sigma}} B \subset W^{\kappa_t, p}(\mathbb{R}; W^{\kappa_x, p}(\mathbb{R}^d)),$$

for  $\kappa_t < \sigma_t$ ,  $\kappa_x < \sigma_x$ .



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Optimal regularity for the porous medium equation.

[arXiv:1708.04408](https://arxiv.org/abs/1708.04408); to appear in JEMS, 2019+.



B. Gess, J. Sauer, and E. Tadmor.

Optimal regularity in time and space for the porous medium equation.

[arXiv:1902.08632 \[math\]](https://arxiv.org/abs/1902.08632), Feb. 2019.

*Happy Birthday, Franco!*