Fluctuations, SPDEs and fluids

Benjamin Gess

Universität Bielefeld & Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig

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joint work with Ben Fehrman [Invent. Math. 2023+]

and Daniel Heydecker [arxiv]

Aim:

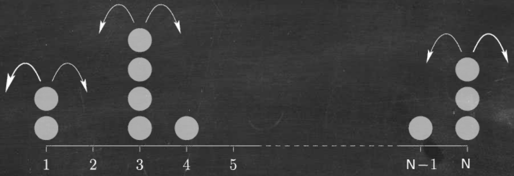
- How to correct for thermal fluctuations in Navier-Stokes?
- Generally: How to correct for fluctuations around PDEs?

Content:

- Toy model: Zero range process and (nonlinear) diffusion equations
  - Relevance of fluctuations
  - Gradient flows and fluctuation-dissipation principle - SPDEs
  - Large deviations and PDEs with irregular coefficients
- Landau-Lifschitz Navier-Stokes equations, and their large deviations

How to correct for fluctuations around PDEs? From interacting particle systems to conservative SPDEs

The zero range process (could also consider simple exclusion, independent particles, ..).



- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N}$, i.e. configurations $\eta : \mathbb{T}_N \to \mathbb{N}_0$: System in state η if container k contains $\eta(k)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \to \mathbb{R}_0^+$.
- Translation invariant, asymmetric, zero mean transition probability

$$p(k,l) = p(k-l), \quad \sum_k kp(k) = 0.$$

- Markov jump process $\eta(t)$ on $\mathbb{M}_N$.
- $\eta(k,t) =$ number of particles in box k at time t.

- **Hydrodynamic limit?** Multi-scale dynamics

- Empirical density field: $\mu^N(x,t) := \frac{1}{N}\sum_k \delta_{\frac{k}{N}}(x)\eta(k,tN^2)$.
- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\,dx$$

with

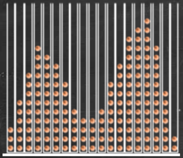
$$\partial_t \bar{\rho} = \partial_{xx}\Phi(\bar{\rho})$$

with Φ the mean local jump rate $\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))]$.

- Loss of information:
  - ▶ Model / Approximation error: $\mu^N = \bar{\rho} + O(N^{-\frac{1}{2}})$
  - ▶ Fluctuations, rare events / large deviations?

Fluctuating Hydrodynamics?

Microscopic scale: Particles

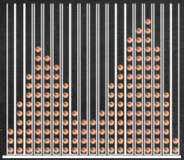


Gridsize = $\frac{1}{N}$

Macroscopic scale: PDEs

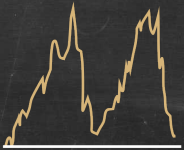
Mean dynamics

Microscopic scale: Particles



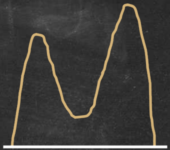
Gridsize = $\frac{1}{N}$

Mesoscopic scale: Conservative SPDEs



Fluctuation correction

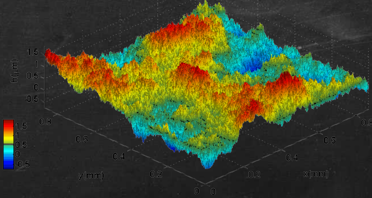
Macroscopic scale: PDEs



Mean dynamics

Ansatz: Conservative SPDEs

$$\partial_t \rho^N = \partial_{xx}\Phi(\rho^N) + N^{-\frac{1}{2}}\partial_x\left(\Phi^{\frac{1}{2}}(\rho^N)\xi^N\right),$$



with ξ^N noise, spatially correlated with decorrelation length $\frac{1}{N}$, and white in time,

Rate of convergence?

- Higher order expansion / fluctuation correction: Ansatz

$$\mu^N = \bar{\rho} + \frac{1}{N^{\frac{1}{2}}} Y^1 + \frac{1}{N} Y^2 + ...$$

  What are Y^i?

- [Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988]: Fluctuation density fields

$$\begin{aligned} Y^{1,N}(x,t) &= N^{\frac{1}{2}}(\mu^N(x,t) - \mathbb{E}\mu^N(x,t)) \\ &\approx N^{\frac{1}{2}}(\mu^N(x,t) - \bar{\rho}) \end{aligned}$$

  Then, $\mathcal{L}(Y^{1,N}) \rightharpoonup^* \mathcal{L}(Y^1)$ for $N \to \infty$ with Y^1 the (Gaussian) solution to

$$dY^1(x,t) = \partial_{xx}(\Phi'(\bar{\rho}(x,t))Y^1(x,t))\,dt + \partial_x(\Phi^{\frac{1}{2}}(\bar{\rho}(x,t))\xi)$$

  with ξ space-time white noise.

- Analogously

$$\rho^N = \bar{\rho} + \frac{1}{N^{\frac{1}{2}}} Y^1 + \frac{1}{N} Y^2 + ...$$

- Therefore, $\quad d(\mu^N, \rho^N) = O(N^{-1}) << O(N^{-\frac{1}{2}}) = d(\mu^N, \bar{\rho}).$

Rare events: (Im-)probability to observe a fluctuation ρ. For SDEs:

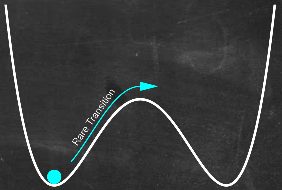
$$dX_t^N = b(X_t^N)dt + N^{-\frac{1}{2}}\sigma(X_t^N)dW_t$$

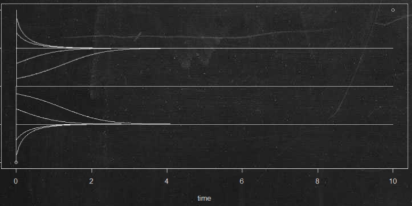
In the limit $N \to \infty$ we have that $X^N \to X^0$ a.s. in $C([0, T]; \mathbb{R}^n)$ with

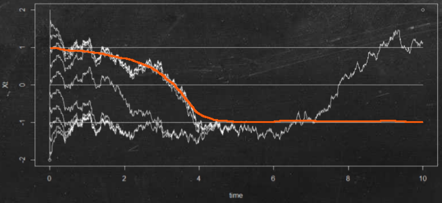
$$dX_t^0 = b(X_t^0)dt.$$

However, for $T \to \infty$ this is generally not true, due to large deviations. For example

$$dX_t^N = -\nabla V(X_t^N)dt + N^{-\frac{1}{2}}dW_t$$
$$V(x) = |x|^4 - |x|^2$$







Freidlin-Wentzel large deviations for SDEs:

$$dX_t^N = b(X_t^N)dt + N^{-\frac{1}{2}}\sigma(X_t^N)dW_t.$$

We get $\mathbb{P}[X^N \approx \rho] = e^{-NI(\rho)}$ with

$$\begin{aligned} I(\rho) &= \inf\{\int_0^T |g|^2 dt : \frac{d}{dt}\rho = b(\rho) + \sigma(\rho)g\} \\ &= \int_0^T (\frac{d}{dt}\rho - b(\rho))(\sigma\sigma^*)^{-1}(\rho)(\frac{d}{dt}\rho - b(\rho))dr \\ &= \|\frac{d}{dt}\rho - b(\rho)\|^2_{(\sigma\sigma^*)^{-1}(\rho)}. \end{aligned}$$

Zero range process [Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995]: (more see later)

$$I(\rho) = \inf\left\{\int_0^T\int_{\mathbb{T}} |g|^2 dxds : g \in L^2_{t,x}, \underbrace{\partial_t\rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)g)}_{\text{"skeleton equation"}}\right\}$$

$$\inf\{\underbrace{\|H\|^2_{H^1_{\Phi(\rho)}}}_{=\int_{t,x}|\partial_x H|^2\Phi(\rho)} : \underbrace{\partial_t\rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)}_{\text{"controlled nonlinear Fokker-Planck equation"}}\}$$

$$= \|\partial_t\rho - \partial_{xx}\Phi(\rho)\|^2_{H^{-1}_{\Phi(\rho)}}.$$

Contraction principle *in large deviation theory: Let* X, Y *be Hausdorff topological spaces and* $f : X \to Y$ *continuous. Let* μ^ε *a sequence of probability measure satisfying a large deviation principle with (good) rate function* I*, that is, for all* $\mathcal{U}, \mathcal{O} \subseteq X$ *closed, open resp. we have that*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon(\mathcal{U}) \leq - \inf_{u \in \mathcal{U}} I(u)$$
$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon(\mathcal{O}) \geq - \inf_{u \in \mathcal{O}} I(u),$$

then $f_* \mu^\varepsilon$ *satisfies an LDP with good rate function*

$$I'(y) := \inf\{I(x) : f(x) = y\}.$$

Informally, correct large deviations:

- Recall
$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + N^{-\frac{1}{2}}\partial_x\left(\Phi^{\frac{1}{2}}(\rho^N)\xi^N\right).$$

- Rare events: (Im-)probability to observe a fluctuation ρ:
$$\mathbb{P}[\rho^N \approx \rho] = e^{-N\, I(\rho)} \quad N \text{ large}$$

- Informally applying the contraction principle to the solution map
$$F : N^{-\frac{1}{2}}\xi \mapsto \rho$$
yields as a rate function
$$I(\rho) = \inf\{I_\xi(g) : F(g) = \rho\}.$$

- Schilder's theorem for Brownian sheet suggests
$$I_\xi(g) = \int_0^T \int_{\mathbb{T}} |g|^2\, dxdt.$$

- Get
$$I(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2\, dxdt : \partial_t \rho = \partial_{xx}\left(\Phi(\rho)\right) + \partial_x\left(\Phi^{\frac{1}{2}}(\rho) g\right)\right\}.$$

A general framework: fluctuating gradient flows

Nonequilibrium statistical mechanics - fluctuating gradient flows

- Many physical systems can be described by a competition between the relaxation of an energy E and friction in terms of a mobility M
- Gradient flow on an (infinite dimensional) "Riemannian manifold" [Jordan, Kinderlehrer, Otto, 1998]

$$\partial_t \rho = -M(\rho)\frac{\partial E}{\partial \rho}(\rho). \qquad dX_t = -M(X_t)\nabla E(X_t)dt.$$

- Nonlinear diffusion, with $\frac{\partial E}{\partial \rho}(\rho) = \log \Phi(\rho)$,

$$\partial_t \rho = \Delta \Phi(\rho) = \nabla \cdot (\Phi(\rho)\nabla \log(\Phi(\rho))) = \overbrace{\partial_x(\Phi(\rho)\partial_x}^{-M(\rho)} \frac{\partial E}{\partial \rho}(\rho))$$

- Formal non-equilibrium stationary Gibbs state $\mu = \frac{1}{Z}e^{-N^{-1}E(\rho)}$.
- Detailed-balance: Fluctuating gradient flow ([Öttinger 2005], fluctuating hydrodynamics [Spohn 1991])

$$\partial_t \rho = -M(\rho)\frac{\partial E}{\partial \rho}(\rho) + N^{-\frac{1}{2}} M^{\frac{1}{2}}(\rho)\xi. \quad dX_t = -M(X_t)\nabla E(X_t)dt + N^{-\frac{1}{2}} M^{\frac{1}{2}}(X_t)dW_t.$$

- Decorrelation length $\approx$ typical particle distance / grid-size: $\xi \to \xi^\delta$
- For example

$$\partial_t \rho = \Delta \Phi(\rho) + \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)\,\xi^\delta)$$

Fluctuating hydrodynamics and macroscopic fluctuation theory

[Spohn 1991] [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim 2015]

Recall: Fluctuating gradient flow

$$\partial_t \rho = -M(\rho)\frac{\partial E}{\partial \rho}(\rho) + N^{-\frac{1}{2}} M^{\frac{1}{2}}(\rho)\xi.$$

Rare events: (Im-)probability of a fluctuation ρ in small noise $N^{-\frac{1}{2}} \to 0$ limit

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\,I(\rho)} \quad N \text{ large.}$$

Informally, "by" contraction principle LDP has rate function

$$I(\rho) = \inf\{\int_{t,x} g^2 : \partial_t \rho = -M(\rho)\frac{\partial E}{\partial \rho}(\rho) + M^{\frac{1}{2}}(\rho) g\}$$

$$= \|M^{-\frac{1}{2}}(\rho)(\partial_t \rho + M(\rho)\frac{\partial E}{\partial \rho}(\rho))\|_{L^2}^2$$

MFT: Use postulated $I(\rho)$ as energy for non-equilibrium systems.

Nonlinear diffusion

$$I(\rho) = \inf\left\{\int_{t,x} g^2 : \partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x\left(\Phi^{\frac{1}{2}}(\rho) g\right)\right\}.$$

Note: This (informally) coincides with the true rate function of the ZRP.

Examples

- E.g. symmetric simple exclusion process:
  [Giacomin, Lebowitz, Presutti, 1999]

$$\partial_t \rho = \Delta \rho + \sqrt{\varepsilon} \nabla \cdot (\sqrt{\rho(1-\rho)}\, \xi^\delta).$$

- More generally (GENERIC):

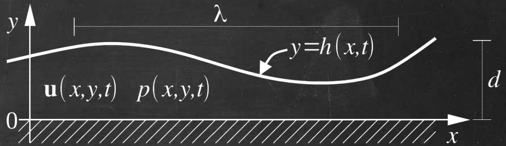
$$\partial_t \rho = \Delta \Phi(\rho) + \nabla \cdot \nu(\rho) + \sqrt{\varepsilon} \nabla \cdot (\sigma(\rho)\, \xi^\delta).$$

- GENERIC: Fluctuating incompressible Navier-Stokes-Fourier (see later)

$$\partial_t v = \Delta v + (v \cdot \nabla) v + \sqrt{\varepsilon} \nabla \cdot (\xi).$$

Stochastic thin films

$$\partial_t h = -\mathrm{div}(h^m \nabla \Delta h) + \mathrm{div}(h^{m/2}\xi)$$



(source: Grün, Mecke, Rauscher, 2006)

Fluctuating gradient flow structure

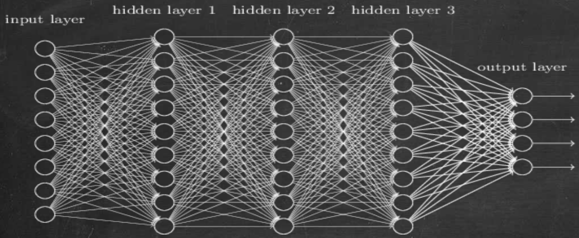
$$\partial_t h = -M(\rho)\frac{\partial E}{\partial h}(h) + M^{\frac{1}{2}}(h)\xi,$$

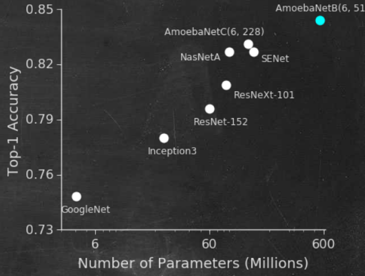
with $E(h) = \frac{1}{2}\int |\nabla h|^2 dx$ and $M(h) = \mathrm{div}(h^m \nabla \cdot)$,
Relevance to include thermal fluctuations:

- Improved prediction of empirical film rupture time scales
  [Grün, Mecke, Rauscher 2006]
- Corrected spreading rate of droplets
  [Davidovitch, Moro, Stone 2005]

Machine learning

Feed-forward neural network





Collecting all parameters $\theta = (\theta_1, \ldots, \theta_M) \in \mathbb{R}^M$

Stochastic gradient descent / empirical risk minimization

$$\theta_{n+1} = \theta_n - \eta \nabla_\theta l(\theta_n, \omega_n),$$

Scaling limits: Small learning rate η, overparametrization $M \to \infty$.

Empirical distribution $\mu_t^M := \frac{1}{M}\sum_i \delta_{\theta_t^i} \to \mu_t$ solution to

$$\partial_t \mu_t = \mathrm{div}(\nabla V(\mu_t, \cdot)\mu_t) + D^2 : (A(\mu_t, \cdot)\mu_t) + \sqrt{\sigma}\mathrm{div}(T(\mu_t, \cdot)\mu\,\xi),$$

where ξ is space-time white noise and V, A, T_μ are non-local operators, see [Chen, Rotskoff, Bruna, Vanden-Eijnden, 2020].

From large deviations to parabolic-hyperbolic PDE with irregular drift

Recall: Rare events are the (im-)probability to observe a fluctuation ρ:

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\, I(\rho)} \quad N \text{ large}$$

A bit more precisely, for every $\mathcal{U}, \mathcal{O}$ closed, open sets

$$\mathbb{P}[\mu^N \in \mathcal{U}] \lesssim e^{-N \inf_{\rho\in\mathcal{U}} I(\rho)}$$

$$e^{-N \inf_{\rho\in\mathcal{O}} I(\rho)} \lesssim \mathbb{P}[\mu^N \in \mathcal{O}]$$

Zero range process

$$I(\rho) = \inf\{\int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt \;: \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho) g)}_{\text{"skeleton equation"}}\}.$$

Theorem ([Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995])

For every $\mathcal{U}, \mathcal{O} \subseteq D([0,T], \mathcal{M}_+)$ *closed, open sets resp. we have*

$$\mathbb{P}[\mu^N \in \mathcal{U}] \lesssim e^{-N \inf_{\rho \in \mathcal{U}} I(\rho)}$$

$$e^{-N \inf_{\rho \in \mathcal{O}} \overline{I_{|A}}(\rho)} \lesssim \mathbb{P}[\mu^N \in \mathcal{O}]$$

where A *is the set of nice fluctuations* $\mu = \rho\, dx$ *with* ρ *a solution to*

$$\partial_t \rho = \partial_{xx} \Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho) g)$$

for some $\underline{g \in C^{1,3}_{t,x}}$.

This is a frequently observed problem: E.g. Fluctuations around Boltzmann equation [Rezakhanlou 1998], [Bodineau, Gallagher, Saint-Raymond, Simonella 2020]. Counter-examples for Boltzmann [Heydecker; 2021].

Problem:

$$I = \overline{I_{|A}}\,?$$

Difficult: Open problem for the zero range process since [Benois, Kipnis, Landim; 1995].

Skeleton equation

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\underbrace{g}_{\in L^2_{t,x}}).$$

How difficult is the well-posedness?

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Scaling and criticality of the skeleton equation

Consider, $\Phi(\rho) = \rho^m$,

$$\partial_t \rho = \partial_{xx}\rho^m + \partial_x(\rho^{\frac{m}{2}} g)$$

with $g \in L^q_t L^p_x$ and $\rho_0 \in L^r_x$.

Via rescaling (“zooming in”):

- $p = q = 2$ is critical.
- $r = 1$ is critical, $r > 1$ is supercritical.

Literature:

- [Le Bris, Lions; CPDE 2008]

$$\partial_t \rho = \frac{1}{2} D^2 : (\sigma\sigma^* \rho) + \mathrm{div}(\rho\, g)$$

  needs $g \in W^{1,1}_{loc,x}$, $\mathrm{div}\, g \in L^\infty$, $\sigma^* \nabla \rho \in L^2_{t,x}$.

- [Karlsen, Risebro; 2003], [Wang, Wang, Li; 2013], [Barbu, Röckner; 2021], roughly speaking

$$\partial_t \rho = \Delta \Phi(\rho) + \mathrm{div}(\Psi(\rho) g)$$

  for $g \in W^{1,1}_{loc,x}$, $\mathrm{div}\, g \in L^\infty$, Ψ locally Lipschitz.

Overview of ingredients of the proof:

- **Part 1:** Apriori-bounds; entropy-entropy dissipation estimates
- **Part 2:** Extending the concepts of DiPerna-Lions, Ambrosio, Le Bris-Lions to nonlinear PDE (but going beyond).
- **Part 3:** Uniqueness for renormalized entropy solutions (variable doubling): New treatment of kinetic dissipation measure. Exploit finite *singular* moments.

Part 1: Apriori-bounds

- Consider

$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \tag{*}$$

  with $g \in L^2_{t,x}$, $m \in [1, \infty)$. E.g.

$$\partial_t \rho = \Delta \rho + \mathrm{div}(\rho^{\frac{1}{2}} g).$$

- Use entropy-entropy dissipation: Evolution of entropy given by $\int_{\mathbb{T}^d} \log(\rho)\rho$. Informally gives

$$\int_x \log(\rho)\rho \,\Big|_0^t + \int_0^t \int_x (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_x g^2.$$

- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting $(*)$ as

$$\begin{aligned}\partial_t \rho &= 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g) \\ &= \mathrm{div}(\rho^{\frac{m}{2}} (\underbrace{2\nabla \rho^{\frac{m}{2}}}_{\in L^2_{t,x}} + \underbrace{g}_{\in L^2_{t,x}}))\end{aligned}$$

- Stability in the control: for $g^\varepsilon \rightharpoonup g$ in $L^2_{t,x}$ by compactness $\rho^\varepsilon \to \hat{\rho}$ weak solution to $(*)$.
- Conclusion: Have to prove uniqueness within this class of solutions.

Part 2: Renormalization
Recall: The linear setting,
[DiPerna, Lions, Invent. Math. 1989; Ambrosio Invent. Math. 2004]

$$\partial_t \rho = \mathrm{div}(\rho g).$$

Then ρ is a renormalized solution, if for all smooth f we have

$$\partial_t f(\rho) = \mathrm{div}(f(\rho)g) - (f(\rho) - f'(\rho)\rho)\mathrm{div} g.$$

For two solutions ρ^1, ρ^2 let $\rho = \rho^1 - \rho^2$. If ρ is renormalized, using, by approximation $f(\rho) = |\rho|$ we get

$$\partial_t \int_x |\rho| = \int_x \mathrm{div}(|\rho| g) - (|\rho| - \mathrm{sgn}(\rho)\rho)\mathrm{div} g = 0.$$

This can be made rigorous for $g \in BV(\mathbb{R}^d)$ (for renormalization), $\mathrm{div} g \in L^\infty(\mathbb{R}^d)$ (for existence).

Let ρ be a weak solution to

$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d.$$

Show that every weak solution is a kinetic solution
(conjoining renormalization [DiPerna, Lions; Ambrosio] with kinetic solutions [Lions, Perthame, Tadmor, J. Amer. Math. Soc. 1994]).
Let

$$\chi(t,x,\xi) = f_\xi(\rho(x,t)) = 1_{0<\xi<\rho(x,t)} - 1_{\rho(x,t)<\xi<0}.$$

Then, informally,

$$\partial_t \chi = m\xi^{m-1}\Delta_x \chi - g(x,t)(\partial_\xi \xi^{\frac{m}{2}})\nabla_x \chi + (\nabla_x g(x,t))\xi^{\frac{m}{2}}\partial_\xi \chi + \partial_\xi q$$

with p parabolic defect measure

$$q = \delta(\xi - \rho(x,t))|\nabla \rho^{\frac{m+1}{2}}|^2.$$

- How to make that rigorous? Take convolution
$$\rho^\varepsilon = \varphi^\varepsilon *_x \rho.$$
- Commutator errors,

$$\begin{aligned}
\partial_t \rho^\varepsilon = \varphi^\varepsilon * \partial_t \rho &= \varphi^\varepsilon * (\Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g)) \\
&= \Delta(\varphi^\varepsilon * \rho^m) + \mathrm{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) \\
&= \Delta(\rho^\varepsilon)^m + \mathrm{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\
&+ \Delta(\varphi^\varepsilon * \rho^m) - \Delta(\rho^\varepsilon)^m \\
&+ \mathrm{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g) - \mathrm{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\
&+ \mathrm{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) - \mathrm{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g).
\end{aligned}$$

- Note: Additional commutator errors by commuting convolution and nonlinearities!
- Commutator estimate using non-standard (optimal) regularity $\rho^{\frac{m}{2}} \in L^2_t \dot{H}^1_x$
- Additional renormalization step to compensate low time integrability $\rho^{\frac{m}{2}} g \in L^1_t L^1_x$.

Theorem

A function $\rho \in L_t^\infty L_x^1$ is a weak solution to

$$\partial_t \rho = 2div(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + div(\rho^{\frac{m}{2}} g)$$

if and only if ρ is a renormalized entropy solution (kinetic solution).

Part 3: Uniqueness for renormalized entropy solutions (variable doubling)

- Established arguments [Chen, Perthame; 2003] not applicable.
- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x,\xi,t) = c_m \delta(\xi - \rho(x,t)) |\nabla \rho^{\frac{m+1}{2}}|^2 = c_m \delta(\xi - \rho(x,t)) \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2$$

does not satisfy

$$\lim_{|\xi|\to\infty} \int_{t,x} q(x,\xi,t)\, dxdt = 0.$$

- Only finite singular moment

$$\int_{t,x,\xi} |\xi|^{-1} q(x,\xi,t)\, d\xi dxdt < \infty.$$

Theorem (The skeleton equation, Fehrman, G. 2022)

Let $g \in L^2_{t,x}$, ρ_0 non-negative and $\int \rho_0 \log(\rho_0)dx < \infty$. There is a unique weak solution to

$$\partial_t \rho = \Delta\Phi(\rho) + \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)g).$$

The map $g \mapsto \rho$, $L^2_{t,x} \to L^1_{t,x}$, is weak-strong continuous. E.g. including all $\Phi(\rho) = \rho^m$, $m \in [1, \infty)$.

Theorem (LDP for zero range process, G., Heydecker, 2023)

The rescaled zero range process satisfies the full large deviations principle with rate function

$$I(\rho) = \frac{1}{2}\|\partial_t \rho - \partial_{xx}\Phi(\rho)\|^2_{H^{-1}_{\Phi(\rho)}}.$$

Content:

- Toy model: Zero range process and (nonlinear) diffusion equations
  - Relevance of fluctuations
  - Gradient flows and fluctuation-dissipation principle - SPDEs
  - Large deviations and PDEs with irregular coefficients
- Landau-Lifschitz Navier-Stokes equations, and their large deviations

Thermal fluctuations in fluids: Landau-Lifschitz-Navier-Stokes

Real fluids have small (thermodynamics) fluctuations on small scales.

Central question: What should the fluctuations (large deviations) around (NSE) be?

Landau-Lifschitz: Physical arguments suggest to add a noise term $-\sqrt{\epsilon}\nabla \cdot \xi$, for a divergence-free white noise ξ. That leads to

$$\partial_t u = \Delta u - \mathrm{P}((u \cdot \nabla)u) - \sqrt{\epsilon}\nabla \cdot \xi, \quad \mathrm{div}(u) = 0$$

GENERIC & fluctuation dissipation:

$$\partial_t u = \overbrace{\Delta}^{=:M(u)} \frac{\partial E}{\partial u}(u) - \mathrm{P}((u \cdot \nabla)u) - \sqrt{\epsilon} \overbrace{\nabla}^{=M^{\frac{1}{2}}(u)} \cdot \xi, \quad \mathrm{div}(u) = 0,$$

with $E(u) = \frac{1}{2}\int |u|^2 dx$. `Gibbs` measure is $\frac{1}{Z} e^{-\varepsilon^{-1}E(u)} du$.

Quastel and Yau: Derivation of Leray-Hopf solutions from a `stochastic` lattice gas & large deviations.

Question: How are they related?

Quastel and Yau [Ann. Math. 1998]: stochastic lattice gas
Number of particles at position x with velocity v is $\eta(x,v) \in \{0,1\}$. Dynamics are given as a joint exclusion process with average velocity v, and a collision operator. Get a pure jump Markov process $\eta(x,v,t)$. The momentum is

$$I_i(\eta) = \sum_v v_i \eta(x,v) \quad i = 1,\dots,d.$$

with empirical momentum density

$$\mu_i(t,dy) = \varepsilon^{d-1} \sum_x \delta_{\varepsilon x}(dy) I_i(\eta)(x, \varepsilon^{-2} t).$$

We then let P_ε be the distribution of the trajectories $\mu_i(t,dy) \in L^2([0,T]; H^{-2})$.

Theorem (Quastel, Yau; 1998)

Assume that the initial conditions $\eta(0,\cdot)$ are in local equilibrium with slowly varying function $u_0 \in L^2$, $d \geq 3$. Then P_ε is tight in $\mathcal{P}(L^2(0,T;H^{-2}))$ with weak topology, and any limit point P is supported entirely on weak solutions to the incompressible Navier-Stokes equations.

Fluctuations: Let

$$\mathbb{X} := L^2_t H \cap C_t(L^2_x)_{\mathrm{w}} \cap (L^2_t \dot{H}^1_x)_{\mathrm{w}}.$$

and

$$I(u) = \frac{1}{2}\|\partial_t u + P(u \cdot \nabla u) - \Delta u\|^2_{L^2(0,T;V')}.$$

Let the set of nice fluctuations be given by

$$A_0 := \{u : \|u\|_{L^\infty(0,T;H)} < \infty, \|u\|^2_{L^2(0,T;V)} < \infty, \|\partial_t u\|_{L^2(0,T;V')} < \infty, \|P(u \cdot \nabla u)\|_{L^2(0,T;V')}$$

Definition

The I-closure of a set $A_0 \subset \mathbb{X}$ is defined by

$$A = \overline{A_0}^{I} := \left\{u \in \mathbb{X} : \exists u^{(n)} \in A_0, \quad u^{(n)} \to u, \qquad I(u^{(n)}) \to I(u)\right\} \tag{1}$$

so that A is the maximal set on which I agrees with the lower semicontinuous envelope of its restriction to A_0.

Theorem (Quastel, Yau; 1998)

With the static rate function $I_{stat}(u_0)$ the total rate function is

$$\mathcal{I}(u) = I_{stat}(u(0,\cdot)) + I(u).$$

Then, P_ε has the restricted large deviations principle

$$\limsup_{\varepsilon\to 0} \varepsilon^{d-2} \log \mathbb{P}_\varepsilon(\mathcal{U}) \leq - \inf_{u\in\mathcal{U}} \mathcal{I}(u)$$
$$\liminf_{\varepsilon\to 0} \varepsilon^{d-2} \log \mathbb{P}_\varepsilon(\mathcal{O}) \geq - \inf_{u\in\mathcal{U}\cap A} \mathcal{I}(u).$$

Informal large deviations for Landau-Lifschitz-Navier-Stokes: Pretend that the contraction principle is applicable to

$$\partial_t u = \Delta u - \mathrm{P}((u \cdot \nabla)u) - \sqrt{\epsilon}\nabla \cdot \xi, \quad \mathrm{div}(u) = 0.$$

Informally applying the contraction principle to the solution map $F : \sqrt{\epsilon}\xi \mapsto u$ yields as a rate function

$$I(u) = \inf\{I_\xi(g) : F(g) = u\}.$$

Schilder's theorem for Brownian sheet: $I_\xi(g) = \frac{1}{2}\int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt$. Get

$$\begin{aligned} I(u) &= \frac{1}{2}\inf\left\{\int_0^T \int_{\mathbb{T}^d} |g|^2 \, dxdt : \partial_t u = \Delta u - \mathrm{P}((u \cdot \nabla)u) - \nabla g\right\} \\ &= \frac{1}{2}\|\partial_t u - \Delta u + \mathrm{P}((u \cdot \nabla)u)\|^2_{L^2(0,T;V')}. \end{aligned}$$

How to make any of this rigorous?

Recall

$$\partial_t u = \Delta u - \mathrm{P}((u \cdot \nabla)u) - \sqrt{\epsilon}\nabla \cdot \xi, \quad \mathrm{div}(u) = 0.$$

*Super*critical SPDE, and cannot be renormalised: Replace $\xi \mapsto \xi^\delta$, correlations on length $\delta \ll 1$ and consider the joint scaling limit $\delta = \delta(\varepsilon) \to 0$.

Proposition

For any $\epsilon, \delta > 0$, there exists a weak Leray-Hopf solution to

$$\partial_t u^{\varepsilon,\delta} = \Delta u^{\varepsilon,\delta} - \mathrm{P}((u^{\varepsilon,\delta} \cdot \nabla)u^{\varepsilon,\delta}) - \sqrt{\epsilon}\nabla \cdot \xi^\delta, \quad \mathit{div}(u^{\varepsilon,\delta}) = 0.$$

Main Result: (restricted) LDP: Challenge for proving an LDP: No uniqueness for the LLNSE is known.

Note: No nontrivial LDP for strong solutions

Theorem (G.-Heydecker-Wu)

Under a scaling relation on ϵ, δ, for any closed $\mathcal{U} \subset \mathbb{X}$, $\mathcal{O} \subset \mathbb{X}$ open

$$\limsup_{\epsilon \to 0} \log \mathbb{P}\left(u^{\varepsilon,\delta} \in \mathcal{U}\right) \leq - \inf_{u \in \mathcal{U}} \mathcal{I}(u)$$

$$\limsup_{\epsilon \to 0} \log \mathbb{P}\left(u^{\varepsilon,\delta} \in \mathcal{O}\right) \geq - \inf_{u \in \mathcal{O} \cap \mathcal{C}} \mathcal{I}(u),$$

where $\mathcal{C}_0$ is any weak-strong uniqueness class and $\mathcal{C}$ is its $\mathcal{I}$-closure.

Definition

We say that $\mathcal{C}_0 \subset \mathbb{X}$ satisfies the weak-strong uniqueness property if, whenever $u \in C_0, v \in \mathbb{X}$ are weak solutions to the skeleton equation

$$\partial_t u = \Delta u - \mathrm{P}((u \cdot \nabla)u) - \nabla \cdot g, \quad \mathrm{div}(u) = 0$$

for the same g and the same initial data $u(0) = v(0)$, and v satisfies the energy inequality

$$\frac{1}{2}\|v(t)\|_H^2 + \int_0^t \|v(s)\|_V^2 ds \leq \frac{1}{2}\|v_0\|_H^2 + \int_0^t \langle \nabla v, g \rangle ds \tag{2}$$

then $u = v$.

E.g. $\mathcal{C}_0 = L^4_{t,x}$.

LDP: Interpretation

The rate function is the same one found by Quastel-Yau:

Links lattice gas to SPDE, see (Dirr-Fehrman-Gess, 19), (Fehrman-Gess, 21), (Gess-Heydecker, 23).

We can see (LLNS) as a numerical method for the lattice gas with nicer sample paths ($U_\varepsilon^\delta \in \mathbb{X}$)...

or, if we view (LLNS) as the physical equation, this shows the physicality of the lattice-gas MFT.

From large deviations and weak-strong uniqueness to the energy equality:

Literature: Several conditions known that give weak-strong uniqueness, e.g. LPS condition

$$u \in L_t^q L_x^p \quad \text{with } \frac{d}{p} + \frac{2}{q} \leq 1,\ p > d.$$

For these conditions it is also known that u satisfies the energy equality.

But: A direct implication between the energy equality and weak-strong uniqueness is not known.

Theorem (From LDP to Energy Equality)

Let $\mathcal{C}$ be the $\mathcal{I}$-closure of a weak-strong uniqueness class $\mathcal{C}_0$ for the forced Navier-Stokes / Skeleton equation. Let $\mathcal{R} \subseteq \mathcal{C}$ and assume that $\mathfrak{T}_T \mathcal{R} = \mathcal{R}$, with $\mathfrak{T}_T u(t,x) := -u(T-t,x)$. Then, $u \in \mathcal{R}$ with finite rate will satisfy the ***energy equality***

$$\frac{1}{2}\|u(T)\|_H^2 + \int_0^T \|\nabla u(t)\|_H^2 dt = \frac{1}{2}\|u(0)\|_H^2 + \int_0^T \langle \nabla u, g \rangle dt.$$

Hence: time-symmetric conditions ($L_t^p L_x^q$, $W_t^{\alpha,p} W_x^{\beta,q}$......) which give weak-strong uniqueness also give the energy equality.

Corollary

Let u be a weak Leray-Hopf solution satisfying weak-strong uniqueness. Assume that $\mathfrak{T}_T u$ satisfies weak-strong uniqueness. Then u satisfies the energy equality.

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