Gradient flow structures and large deviations for porous media equations

Benjamin Gess

TU Berlin & MPI MiS Leipzig

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Stochastic porous medium equation

Stochastic porous medium equation, $\alpha \geq 1$,

$$\partial_t \rho = \Delta \rho^\alpha + \underbrace{\text{noise}}_{?}.$$

Rewrite the PME as a gradient flow

$$\partial_t \rho = \Delta \rho^\alpha =_? -\nabla_{\mathcal{M}} \mathcal{H}(\rho) = -M(\rho)\frac{D\mathcal{H}}{D\rho}(\rho),$$

where $M(\rho)$ the inverse Riemannian tensor, $\mathcal{H}$ some entropy. Choose noise so that $\mu(d\rho) = \frac{1}{Z} e^{-\mathcal{H}(\rho)} d\rho$ becomes an invariant measure, i.e.

$$\partial_t \rho = -M(\rho)\frac{D\mathcal{H}}{D\rho}(\rho) + M^{\frac{1}{2}}(\rho) \diamond \xi.$$

Different gradient flow structures lead to different SPDEs.

Gradient flows for PME:

Brezis ['71]: $\mathcal{M} = H^{-1}$, $M(\rho) = -\Delta$, $\mathcal{H}(\rho) = \int \rho^{\alpha+1}$,

$$\partial_t \rho = \nabla \cdot (\nabla \rho^\alpha).$$

Otto ['01]: $\mathcal{M} = \mathcal{P}(\mathbb{T}^d)$, $M(\rho) = -\nabla \cdot (\rho \nabla \cdot)$, $\mathcal{H}(\rho) = \int \rho^\alpha$ pressure,

$$\partial_t \rho = \nabla \cdot (\rho \nabla \rho^{\alpha-1}).$$

"Thermodynamic metric": $\mathcal{M} = \mathcal{P}(\mathbb{T}^d)$, $M(\rho) = -\nabla \cdot (\rho^\alpha \nabla \cdot)$, $\mathcal{H}(\rho) = \mathcal{H}(\rho)$ Boltzmann entropy,

$$\partial_t \rho = \nabla \cdot (\rho^\alpha \nabla \log(\rho)).$$

Sideremark: Leads to fluctuating hydrodynamics SPDE

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\alpha/2} \diamond \xi).$$

"Thermodynamic metric"? Manifold $\mathcal{M}$ = set of probability measures. Formally, the inverse Riemannian tensor should be

$$M(\rho) = \nabla \cdot \rho^{\alpha} \nabla = \nabla \cdot \rho^{\frac{\alpha}{2}} (\rho^{\frac{\alpha}{2}} \nabla)$$

Following [Otto; 2001], but replacing $\rho \mapsto \rho^{\alpha}$, suggests

$$T_{\rho}\mathcal{M} := \overline{\left\{\rho^{\alpha/2} \nabla \varphi : \varphi \in C^2(\mathbb{T}^d)\right\}}^{L^2_x}$$

with

$$g_{\rho}(\zeta_1, \zeta_2) = \int_{\mathbb{T}^d} \rho^{\alpha/2} \nabla \xi_1 \cdot \rho^{\alpha/2} \nabla \xi_2$$

and

$$\zeta_i + \nabla \cdot (\frac{1}{2} \rho^{\alpha} \nabla \xi_i) = 0.$$

However, this does not lead to a Riemannian metric, since we can have $\rho_0 \neq \rho_1$ with $d(\rho_0, \rho_1) = 0$ (unbounded diffusivity), or $d(\rho_0, \rho_1) = \infty$ (degeneracy). $(\alpha > 1)$.

"Thermodynamic metric": Consider the non-degenerate case

$$\partial_t \rho = \nabla \cdot (\rho \nabla \log(\rho)).$$

Then there is a rigorous meaning for

$$\partial_t \rho = \Delta \rho = -\nabla_{\mathcal{M}} \mathcal{H}(\rho) = -M(\rho)\frac{D\mathcal{H}}{D\rho}(\rho). \qquad (\star)$$

Note

$$\partial_t \mathcal{H}(\rho) = -(\partial_t \rho, \frac{D\mathcal{H}}{D\rho})_{M(\rho)} \geq -|\partial_t \rho|_{M(\rho)} |\frac{D\mathcal{H}}{D\rho}|_{M(\rho)} \geq -\frac{1}{2}|\frac{D\mathcal{H}}{D\rho}|^2_{M(\rho)} - \frac{1}{2}|\partial_t \rho|^2_{M(\rho)}$$

with equality iff ρ solves $(\star)$. $((v,w) = -\frac{1}{2}|v|^2 - \frac{1}{2}|w|^2$ iff $v = -w)$

Consequence: ρ is a gradient flow for $(\star)$ iff

$$0 = I(\rho) = \mathcal{H}(\rho_0) - \mathcal{H}(\rho_T) - \frac{1}{2}\int_0^T \int_x \frac{|\nabla \rho|^2}{\rho}\, dxdt - \frac{1}{2}\mathcal{A}(\rho)$$

where

$$\mathcal{A}(\rho) = \int_0^T |\dot{\rho}|^2_{M(\rho)} = \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t \rho + M^{\frac{1}{2}}(\rho) g = 0\}.$$

is the action of ρ.

Definition: ρ is a "thermodynamic" gradient flow of

$$\partial_t \rho = \nabla \cdot (\rho^\alpha \nabla \log(\rho)).$$

iff

$$0 = \mathcal{I}(\rho) = \mathcal{H}(\rho_0) - \mathcal{H}(\rho_T) - \frac{1}{2}\int_0^T \int \frac{|\nabla \rho^{\frac{\alpha+1}{2}}|^2}{\rho} dxdt - \frac{1}{2}\mathcal{A}(\rho).$$

Gradient flows and large deviations

Rare events are the (im-)probability to observe a fluctuation ρ:

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\mathcal{I}(\rho)} \quad N \text{ large}$$

We say that a gradient flow structure corresponding to an energy $\mathcal{I}$ is thermodynamic, if there is a particle system μ^N satisfying an LDP with rate function $\mathcal{I}$.

Macroscropic Fluctuation Theory [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015].

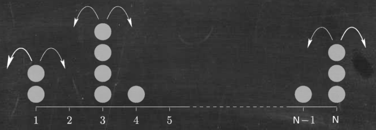
Q: it is not (rigorously) known which, if any, of the gradient flow structures of PME are thermodynamic.

The porous medium equation as a hydrodynamic limit

Can we obtain the PME as a limit of a (stochastic) particle system?

E.g. [Suzuki, Ushiyama; 1993], [Ekhaus, Seppäläinen; 1996], [Oelschläger; 1990], [Gonçalves, Landim, Toninelli; 2009], [Gonçalves, Nahum, Simon; 2023], also fractional cases [Cardoso, de Paula, Gonçalves; 2023]

The zero range process:



Local jump rate function $g(\eta) = \eta^\alpha : \mathbb{N}_0 \to \mathbb{R}_0^+$, $\alpha > 1$.

Translation invariant, asymmetric, zero mean transition probability

$$p(k,l) = p(k-l), \quad \sum_k k p(k) = 0.$$

Generator

$$L_N F(\eta) := \sum_{x,y \in \mathbb{T}_N^d} p(x,y)\eta^\alpha(x)(F(\eta^{x,y}) - F(\eta)).$$

Hydrodynamic limit Empirical density field:

$$\mu^N(x,t) := \Big(\frac{1}{N}\sum_k \delta_k(x)\eta(k,t)\Big)(xN, tN^2).$$

[Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\,dx$$

with

$$\partial_t \bar{\rho} = \Delta\Phi(\bar{\rho})$$

with Φ the mean local jump rate $\Phi(\bar{\rho}) = \mathbb{E}_{\nu_{\bar{\rho}}}[\eta^\alpha(0)]$.

The Φ is non-degenerate: $\Phi' \geq c > 0$. Even if $g(\eta) = \eta^\alpha$ we do not see the porous medium equation, that is, $\Phi(\bar{\rho}) \neq \bar{\rho}^\alpha$, $\alpha \geq 1$.

The porous medium equation, $\alpha \geq 1$,

$$\partial_t \bar{\rho} = \Delta \bar{\rho}^\alpha$$

Scaling invariance of PME: Let $\tilde{\rho}(t,x) = \chi \bar{\rho}(\tau t, \lambda x)$. Then

$$\partial_t \tilde{\rho} = \tau \chi^{1-\alpha} \lambda^{-2} \Delta \tilde{\rho}^\alpha.$$

Get a one parameter family of scaling invariances

$$\tau \chi^{1-\alpha} \lambda^{-2} = 1.$$

Consider the ZRP with local jump rate function $g(\eta) = \eta^\alpha$, $\alpha \geq 1$.

Rescaling particle sizes by χ_N

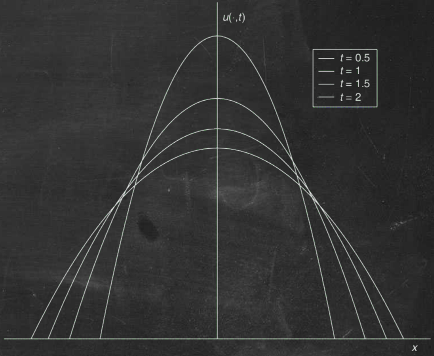
$$\mu^N(x,t) := \chi_N \Big(\frac{1}{N} \sum_k \delta_k(x) \eta(k,t) \Big) (xN, t \frac{N^2}{\chi_N^{1-\alpha}}).$$

Two difficulties:

- Superlinear growth of $g(\eta) = \eta^\alpha$, $\alpha > 1$. Possible concentration of mobility $(\eta^N(x))^\alpha$.
- Degeneracy of $g(\eta) = \eta^\alpha$ at $\eta = 0$. Now becomes visible with χ small. Dirichlet form degenerates.

As a result, the classical one-block, two-block approach to the superexponential replacement lemma is not applicable.

Macroscopic: Barenblatt solution



Solution: New microscopic, "pathwise" entropy-dissipation inequality

Theorem (Hydrodynamic limit, G., Heydecker, 2023)
Let $\rho_0 \in L^1_{\geq 0}(\mathbb{T}^d)$ *with finite entropy* $\mathcal{H}(\rho_0) = \int \rho_0 \log \rho_0 - \rho_0 + 1 < \infty$, $\limsup_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}(\mathcal{H}(\eta_0^N) > M) = 0$ *and*

$$\mathbb{P}(d(\eta_0^N, \rho_0) > \varepsilon) \to 0,$$

where d *is a metric inducing the weak-$*$ topology of* $L^1_{\geq 0}(\mathbb{T}^d)$*. Assume the scaling relation* $\chi_N^{1\wedge\alpha/2} \leq CN^{-2}$*. Then*

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\, dx$$

in probability, where $\bar{\rho}$ *is the solution to*

$$\partial_t \bar{\rho} = \Delta \bar{\rho}^\alpha.$$

"Pathwise" entropy-dissipation inequality: Let $\mathcal{F}_N$ be the following functional on discrete paths

$$\mathcal{F}_N(\eta^N) := \sup_{t\le T} \mathcal{H}(\eta^N_t) + \int_0^T \mathcal{D}_{\alpha,N}(\eta^N_s) ds$$

where $\mathcal{D}_{\alpha,N}$ is a lattice discretisation of $\mathcal{D}_\alpha(\rho) = \int_x |\nabla \rho^{\alpha/2}|^2 \, dx$:

$$\mathcal{D}_{\alpha,N}(\eta^N) := \frac{1}{2\alpha N^{d-2}} \sum_{x\sim y} ((\eta^N(x))^{\alpha/2} - (\eta^N(y))^{\alpha/2})^2.$$

Macroscopically

$$\mathcal{H}(\bar\rho_T) + \int_0^T \int_x |\nabla \bar\rho^{\alpha/2}|^2 dx \le \mathcal{H}(\bar\rho_0).$$

Mesoscopically/SPDE

$$\mathcal{H}(\rho_T) + \int_0^T \int_x |\nabla \rho^{\alpha/2}|^2 dx \le \mathcal{H}(\rho_0) + \text{martingale} + \frac{\chi_N^2 C(N)}{N^d} \text{Ito-correction} \ .$$

Explicit computation, using $\chi_N^{1\wedge\alpha/2} \leq CN^{-2}$ yields that

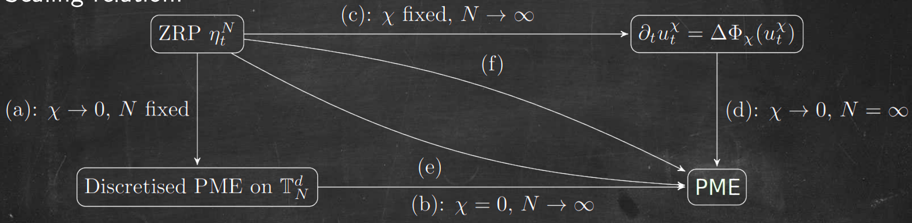
$$Z_t^N = exp\left(\frac{N^d\alpha}{2\chi_N}\left(\mathcal{H}(\eta_t^N) - \int_0^t \mathcal{D}_{\alpha,N}(\eta_s^N)ds - Ct\right)\right)$$

is a supermartingale.
Post-process:

$$\limsup_{M\to\infty}\limsup_N \frac{\chi_N}{N^d}\log\mathbb{P}\left(\mathcal{F}_N(\eta_\cdot^N) > M\right) = -\infty.$$

Implies $L^\beta(\mathbb{T}^d)$-estimate, some $\beta > \alpha$, equicontinuity. One can then conclude by (stochastic) compactness / Aubin-Lions-Simon type theory.

Scaling relation:



Here: Assume the "scaling relation"

$$\chi_N^{1\wedge\alpha/2} \leq CN^{-2}.$$

Large deviations around the porous medium equation?

Rate function

$$I(\rho) = \inf\left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dxds : \ g \in L^2_{t,x}, \underbrace{\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)}_{\text{"skeleton equation"}} \right\}$$

$$= \inf\{ \underbrace{\|H\|^2_{H^1_{\rho^\alpha}}}_{=\int_{t,x} |\nabla H|^2 \rho^\alpha} : \underbrace{\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^\alpha \nabla H)}_{\text{"controlled nonlinear Fokker-Planck equation"}} \}.$$

Theorem ([Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995])

For every $\mathcal{U}, \mathcal{O} \subseteq D([0,T], \mathcal{M}_+)$ closed, open sets resp. we have

$$\mathbb{P}[\mu^N \in \mathcal{U}] \lesssim e^{-N \inf_{\rho \in \mathcal{U}} I(\rho)}$$

$$e^{-N \inf_{\rho \in \mathcal{O}} \overline{I_{|A}}(\rho)} \lesssim \mathbb{P}[\mu^N \in \mathcal{O}]$$

where A is the set of nice fluctuations $\mu = \rho\, dx$ with ρ a solution to

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)$$

for some $\underline{g \in C^{1,3}_{t,x}}$. $\underline{\text{Problem:}}\, I = \overline{I_{|A}}$?

One approach: Show well-posedness of

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g), \quad \text{with } g \in L^2_{t,x}.$$

Theorem (The skeleton equation, Fehrman, G. 2023)

Let $g \in L^2_{t,x}$, ρ_0 non-negative and $\int \rho_0 \log(\rho_0) dx < \infty$. There is a unique weak solution to

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

The map $g \mapsto \rho$, $L^2_{t,x} \to L^1_{t,x}$, is weak-strong continuous.

Theorem (LDP for zero range process, G., Heydecker, 2023)

The rescaled zero range process satisfies the full large deviations principle with speed $\frac{N^d}{\chi_N}$ and rate function

$$I(\rho) = \inf \left\{ \|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \right\}.$$

Gradient flow structures for the porous medium equation

The PME as a gradient flow

$$\partial_t \rho = \Delta \rho^\alpha \mathbin{=_?} -\nabla_{\mathcal{M}} \mathcal{H}(\rho^\alpha) = -M(\rho) \frac{D\mathcal{H}}{D\rho}(\rho),$$

The large deviations select the skeleton equation

$$\partial_t \rho = \Delta \rho^\alpha + \underbrace{\nabla \cdot (\rho^{\alpha/2}}_{=:M^{\frac{1}{2}}(\rho)} g).$$

This suggests

$$\begin{aligned} \partial_t \rho &= \nabla \cdot (\rho^\alpha \nabla \log(\rho)) + \nabla \cdot (\rho^{\alpha/2} g) \\ &= -M(\rho) \frac{D\mathcal{H}}{D\rho}(\rho) + M^{\frac{1}{2}}(\rho) g, \end{aligned}$$

i.e. the "thermodynamic metric".

Obstacle: Have to rewrite rate function in terms of energy.

If we are able to write $\Delta\rho^\alpha = -\nabla_{\mathcal{M}}\mathcal{H}(\rho^\alpha)$ then we have the following identity

$$\begin{aligned}\mathcal{J}(\rho) =& \inf\left\{\int_0^T\int_{\mathbb{T}}|g|^2 dxds : g\in L^2_{t,x},\, \partial_t\rho = \Delta\rho^\alpha + \nabla\cdot(\rho^{\alpha/2}g)\right\}\\ =&\|\partial_t\rho - \Delta\rho^\alpha\|^2_{H^{-1}_{\rho^\alpha}} = \|\partial_t\rho\|^2 - 2(\partial_t\rho, -\nabla_{\mathcal{M}}\mathcal{H}(\rho^\alpha))_{H^{-1}_{\rho^\alpha}} + \|\Delta\rho^\alpha\|^2_{H^{-1}_{\rho^\alpha}}\\ =&\mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\int_0^T \|\partial_t\rho\|^2_{\dot{H}^{-1}_{\rho^\alpha}} + \frac{1}{2}\|\Delta\rho^\alpha\|^2_{\dot{H}^{-1}_{\rho^\alpha}}.\end{aligned}$$

Define the action

$$\mathcal{A}(\rho) = \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t\rho + \underbrace{\nabla\cdot(\rho^{\alpha/2}}_{=M^{\frac{1}{2}}(\rho)} g) = 0\}.$$

Informally

$$\mathcal{A}(\rho) = \int_0^T \|\partial_t\rho\|^2_{\dot{H}^{-1}_{\rho^\alpha}}.$$

In conclusion, the gradient flow picture suggests the energy identity

$$\mathcal{J}(\rho) = \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\mathcal{A}(\rho) + \frac{1}{2}\int_0^T \|\rho^{\alpha/2}\|^2_{\dot{H}^1}.$$

Theorem (Entropy dissipation equality, G., Heydecker, 2023)

Let $D_\alpha(\rho) < \infty$, $\mathcal{H}(\rho_0) < \infty$, $u_0 > 0$. Then

$$\mathcal{J}(\rho) = \mathcal{H}_{u_0}(\rho_T) - \mathcal{H}_{u_0}(\rho_0) + \frac{1}{2}\mathcal{A}(\rho) + \frac{1}{2}\int_0^T \|\rho^{\alpha/2}(s)\|_{\dot{H}^1}^2 \, ds.$$

If ρ is a solution to the PME, we have the energy equality

$$0 = \mathcal{H}_{u_0}(\rho_T) - \mathcal{H}_{u_0}(\rho_0) + \int_0^T \|\rho^{\alpha/2}(s)\|_{\dot{H}^1}^2 \, ds.$$

Sketch of the proof

In equilibrium, detailed balance $\implies$ $(\mathcal{T}\eta^N_\bullet)_t := \eta^N_{T-t-}$ has the same law as the original process.

Contraction principle and uniqueness of rate functions:

$$\mathcal{I}(\mathcal{T}\rho) = \mathcal{I}(\rho)$$

for all ρ. Analyse identity without assuming any more regularity on ρ than necessary.

Recall

$$\mathcal{J}(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2 dxds : \ g \in L^2_{t,x}, \ \partial_t \rho = \Delta\rho^\alpha + \nabla\cdot(\rho^{\alpha/2} g)\right\}$$

$$\mathcal{A}(\rho) = \inf\{\|\theta\|^2_{L^2_{t,x}} : \ \partial_t\rho + \nabla\cdot(\rho^{\alpha/2}\theta) = 0\}.$$

Optimal g, θ are uniquely characterised by membership in

$$\Lambda_\rho := \overline{\left\{\rho^{\alpha/2}\nabla\varphi : \varphi \in C^{1,2}([0,T]\times\mathbb{T}^d)\right\}}^{L^2_{t,x}}.$$

Let $\Pi[\rho]$ be orthogonal projection to this space.

If $\mathcal{I}(\rho) < \infty$, let g be optimal.Since

$$\partial_t\rho = \Delta\rho^\alpha + \nabla\cdot(\rho^{\alpha/2} g) = \nabla\rho^{\alpha/2}\cdot(2\nabla\rho^{\alpha/2} + g)$$

optimal θ is

$$-2\Pi[\rho]\nabla\rho^{\alpha/2} - g.$$

For time reversal $\rho_r := \mathcal{T}\rho$ we have

$$\begin{aligned}\partial_t \rho_r &= -\Delta\rho_r^\alpha + \nabla\cdot(\rho_r^{\alpha/2}\mathcal{T}g)\\ &= \Delta\rho_r^\alpha - \nabla\cdot\rho_r^{\frac{\alpha}{2}}(2\nabla\rho_r^{\frac{\alpha}{2}} - \mathcal{T}g).\end{aligned}$$

So optimal g for ρ_r is

$$g_{\mathrm{r}} := 2\Pi[\rho]\nabla\rho^{\alpha/2} - \mathcal{T}g.$$

We get

$$\begin{aligned}0 &= \mathcal{I}(\mathcal{T}\rho) - \mathcal{I}(\rho)\\ &= \alpha\mathcal{H}(\rho_T) + \frac{1}{2}\|g_r\|^2 - \alpha\mathcal{H}(\rho_0) - \frac{1}{2}\|g\|^2\\ &= \alpha\mathcal{H}(\rho_T) - \alpha\mathcal{H}(\rho_0) + \frac{1}{2}(\|g_r\|^2 + \|\mathcal{T}g\|^2) - \frac{1}{2}\|g\|^2\\ &= \alpha\mathcal{H}(\rho_T) - \alpha\mathcal{H}(\rho_0) + \frac{1}{4}(\|g_r + \mathcal{T}g\|^2 + \|g_r - \mathcal{T}g\|^2) - \frac{1}{2}\|g\|^2\\ &= \alpha\mathcal{H}(\rho_T) - \alpha\mathcal{H}(\rho_0) + \left\|\Pi[\rho]\nabla\rho^{\alpha/2}\right\|^2_{L^2_{t,x}} + \frac{1}{2}\mathcal{A}(\rho) - \frac{1}{2}\|g\|^2.\end{aligned}$$

We get

$$\mathcal{J}(\rho) = \frac{1}{2}\left(\alpha\mathcal{H}(\rho_T) - \alpha\mathcal{H}(\rho_0) + \left\|\Pi[\rho]\nabla\rho^{\alpha/2}\right\|^2_{L^2_{t,x}} + \mathcal{A}(\rho)\right).$$

Since $\|\Pi[\rho]\nabla\rho^{\alpha/2}\|^2_{L^2_{t,x}} \le \frac{\alpha}{2}\int_0^T \mathcal{D}_\alpha(\rho_s)ds$, the previous argument yields the inequality

$$\mathcal{J}(\rho) \le \frac{1}{2}\left(\alpha\mathcal{H}(\rho_T) - \alpha\mathcal{H}(\rho_0) + \frac{\alpha}{2}\int_0^T \mathcal{D}_\alpha(\rho_s)ds + \frac{1}{2}\mathcal{A}(\rho)\right). \qquad (1)$$

If $\mathcal{F}(\rho) < \infty$, then $\nabla\rho^{\alpha/2} = \frac{2}{\alpha}\rho^{\alpha/2}\nabla\log\rho \in \Lambda_\rho$, so both of the inequalities are equalities.
For the general case, use recovery sequences and use (1) again.

Remark

- The same identity as informally suggested in Dirr-Stamatakis-Peletier.
- Sandier-Serfaty (in)equality for the formal Riemannian structure.
- LDP allows us to avoid proving a 'chain rule for entropy' (Erbar, '16).
- Same argument: equality in the *H*-Theorem for (PME):

$$\mathcal{H}(u_t) + \int_0^t \alpha \mathcal{D}_\alpha(u_s) ds = \mathcal{H}(u_0).$$

A new look at properties of the skeleton equation

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^\alpha \nabla H) = \Delta \rho^\alpha + \nabla \cdot (\rho^{\alpha/2} g)$$

- Construction of g_r shows how *anti*dissipative effects can arise, since

$$\begin{aligned} \partial_t \rho_r &= -\Delta \rho_r^\alpha + \nabla \cdot (\rho_r^{\alpha/2} \mathcal{T} g) \\ &= \Delta \rho_r^\alpha - \nabla \cdot \rho_r^{\frac{\alpha}{2}} (g_r). \end{aligned}$$

- Hence why L^p_x estimates had to be false: trajectories with $\rho_0 \notin L^p_x$, $\rho_T \in C^\infty_x$ give reversal $\rho_0 \in C^\infty_x$ but $\rho_T \notin L^p_x$.

References:

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