Qualitative behavior of quasilinear SPDE

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Outline

1. Random dynamical systems approach to SPDE
   - Existence of stochastic flows
2. Regularization by noise
3. Self-organized criticality

Introduction

- Principal aim: Understand the qualitative behavior of stochastic flows induced by SPDE
$$dX_t = A(X_t)dt + B(X_t)dW_t,$$
where A may be an unbounded operator.

What we would like to have:

- existence and uniqueness
- existence/selection of a stochastic semi-flow
- invariant manifolds
- Lyapunov exponents
- attractors

Existence of stochastic flows

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Existence of stochastic flows

- Aim: Apply methods from dynamical systems to flows of solutions to SPDE
- e.g. Oseledets multiplicative ergodic theorem
- For an SDE

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t$$
$$X_0^x = x \in \mathbb{R}^d$$

  this amounts to study the dynamics on $\mathbb{R}^d$.
- In contrast to studying the Markovian dynamics, i.e.

$$P_t f(x) := \mathbb{E} f(X_t^x),$$

  which amounts to dynamics on $\mathscr{M}_1(\mathbb{R}^d)$ by duality.
- Need: flow property:

$$S(t,s;\omega)x = S(t,r;\omega)S(r,s;\omega)x, \forall s \le r \le t$$

  and cocycle property

$$S(t,s;\omega)x = S(t-s,0;\theta_s\omega), \forall s \le r \le t.$$

- Leads to random dynamical systems

Existence of stochastic flows

- For SDE this problem is solved, i.e. there are good criteria when SDE induce random dynamical systems.
- In infinite dimensions there is no general result and no general techniques.

Simple case: Additive noise

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$$dX_t = A(X_t)dt + dW_t.$$

- Then set: $Y := X - W$. We obtain

$$dY_t = A(Y_t + W_t)dt.$$

- Well-posedness of this transformed equation yields the existence of a stochastic flow.
- Extension for general noise, non-degenerate/rough noise, singular drifts: [G., Liu, Röckner; JDE 1012], [G., JDDE 2013], [G., JDE 2013].

The stochastic porous medium equation

- Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain, we consider

$$
\begin{aligned}
dX_t &= \Delta X_t^{[m]} dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \\
X(0) &= X_0,
\end{aligned}
\tag{1}
$$

with $m > 1$ and $r^{[m]} := r^m \mathrm{sgn}(r)$.

- zero Dirichlet boundary conditions
- driving signals $z \in C([0,T];\mathbb{R}^N)$, $f_k \in C^\infty(\bar{\mathcal{O}})$.
- Existence and uniqueness of solutions e.g. [Prévôt, Röckner; 2007]

The stochastic porous medium equation

- Construction of a stochastic flow for

$$dX_t = \Delta X_t^{[m]} dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}, \text{ on } \mathcal{O}_T.$$

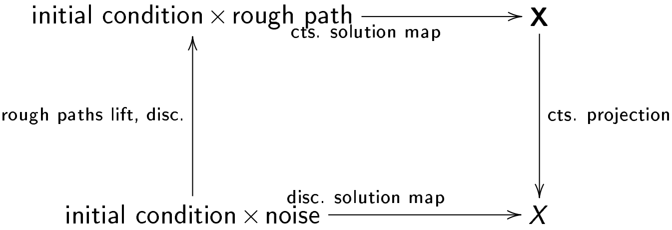
- Let $\mu_t(\xi) := -\sum_{k=1}^N f_k(\xi)\beta_t^{(k)}$. The transformation $Y = e^{\mu} X$ yields:

$$dY_t = e^{\mu_t} \Delta(e^{-m\mu_t} Y_t^{[m]}). \tag{TPME}$$

- Leads to new types of nonlinear PDE
- First well-posedness for (TPME): [Barbu, Röckner; JDE 2011].
- Construction of a continuous random dynamical system (in $L^1(\mathcal{O})$) in [G.; AoP, 2013].

Promising approaches

- Rough paths:



- Stochastic viscosity solutions [Lions, Souganidis; ∼1998].

Regularization by noise

Regularization by noise

Regularization by noise

- Several regularizing effects are known: Well-posedness by noise, regularity of solutions, ergodicity, ...
- here: stabilization or synchronization by noise, i.e. simplification of the long-time behavior of the stochastic flow associated to SPDE by noise.
- Model case: Chafee-Infante equation

$$dX_t = \Delta X_t dt + (\lambda X_t - X_t^3)dt, \quad \lambda \geq 0, \qquad \text{(RDE)}$$

with zero Dirichlet boundary conditions on bounded domains $\mathcal{O} \subseteq \mathbb{R}^d$.

Chafee-Infante equation

- Linearization of Chafee-Infante:

$$dX_t = \Delta X_t dt + \lambda X_t dt$$

- Expansion of X_t in eigenvectors e_k of $-\Delta$: $X_t = \sum_k X_t^k e_k$ yields

$$dX_t^k = (\lambda - \lambda_k) X_t^k dt$$

- For $\lambda = 0$ the invariant solution $X \equiv 0$ is (exponentially) stable.
- As λ increases 0 bifurcates whenever λ crosses eigenvalues λ_k.
- Consequence: Finite dimensional attractor with explicitly known dimension (for Chafee-Infante).

Chafee-Infante equation

- Synchronization by additive noise: Consider

$$dX_t = \Delta X_t dt + \lambda X_t dt + dW_t,$$

where $W_t = \sum_{k=1}^{\infty} \sigma_k e_k \beta_t^k$.

- Expansion in eigenmodes: $X_t = \sum_k X_t^k e_k$ yields

$$dX_t^k = (\lambda - \lambda_k) X_t^k dt + \sigma_k d\beta_t^k.$$

- Ergodicity for Chafee-Infante if $\sigma_k \neq 0$ for all $\lambda > \lambda_k$ $\rightarrow$ regularization by noise on the level of ergodicity

$$\mathscr{L}(X_t) \rightarrow_{TV} \mu.$$

- In fact: Since comparison holds for Chafee-Infante, one can show

$$\|X_t^x - X_t^y\| \rightarrow 0, \quad \text{for } t \rightarrow \infty,$$

for each $x, y \in H$.

Chafee-Infante equation

- Synchronization by multiplicative noise:

$$dX_t = \Delta X_t dt + \lambda X_t dt + \sum_{k=1}^{\infty} B_k X_t \circ d\beta_t^k,$$

  B_k suitable linear operators. It is enough to have B_k acting on the finite-dimensional unstable part. Under proper assumptions on B_k system becomes ergodic.
- In fact: The invariant 0-solution becomes exponentially stable

Porous medium equations

- PME:

$$\begin{aligned} dX_t &= \Delta X_t^{[m]} dt + \lambda X_t dt \\ &= {X_t}^{m-1} \Delta X_t dt + {X_t}^{[m-2]} |\nabla X_t|^2 dt + \lambda X_t dt. \end{aligned} \qquad \text{(PME)}$$

- Intuitively: Ellipticity constant at the invariant solution $X \equiv 0$ degenerates $\rightarrow$ all modes become unstable.
- [Efendiev, Zelik; 2008]: Attractor of (PME) has infinite fractal dimension for $\lambda > 0$.

Porous medium equations

- Is it possible to obtain synchronization by noise for

$$dX_t = \Delta X_t^{[m]} dt + \lambda X_t dt?$$

- Answer: Yes, by additive noise [G.; JDDE, 2012]

$$dX_t = \Delta X_t^m dt + \lambda X_t dt + dW_t$$

  with W_t sufficiently "strong/non-degenerate" noise.
- Answer: Not by multiplicative noise [G.; SIMA, 2013], i.e.

$$dX_t = \Delta X_t^m dt + \lambda X_t dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^k$$

  has an infinite dimensional random attractor regardless of N.
- A similar analysis seems possible for the stochastic p-Laplace equation

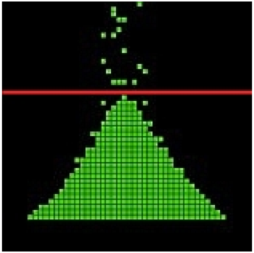
$$dX_t = \mathrm{div}(|\nabla X_t|^{[m-1]} \nabla X_t) dt + \lambda X_t dt.$$

Self-organized criticality

Self-organized criticality

Cellular automata model

- The following model goes back to [Bantay, Ianosi; Physica A, 1992].
- Cellular automata model for a sandpile:



Cellular automata model

- Consider an $N \times N$ square lattice, representing a discrete region $\mathscr{O} = \{(i,j)\}_{i,j=1}^{N}$.
- At each site (i,j) the height of the sandpile at time t is h_{ij}^{t}.
- The system is perturbed externally until the height h exceeds a threshold (critical) value h^c.
- Then, a toppling (avalanche) event occurs: The toppling at any 'activated' site (k,l) is described by:

$$h_{ij}^{t+1} \to h_{ij}^{t} - M_{ij}^{kl}, \quad \forall (i,j) \in \mathscr{O},$$

  where

$$M_{ij}^{kl} = \begin{cases} 4 & (k,l) = (i,j) \\ -1 & (k,l) \sim (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

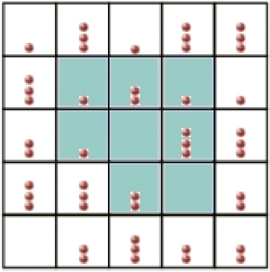
- Rewrite as:

$$h_{ij}^{t+1} - h_{ij}^{t} = -M_{ij}^{kl} H(h_{ij}^{t} - h_{ij}^{c}), \quad \forall (i,j) \in \mathscr{O},$$

  where H is the Heaviside function.

Cellular automata model

- The avalanches are continued until no site exceeds the threshold (which obviously happens after finitely many steps).
- As an example: (animated gif)



Continuum limit

- Passing to a continuum limit in

$$h_{ij}^{t+1} - h_{ij}^{t} = -M_{ij}^{kl} H(h_{ij}^{t} - h_{ij}^{c}), \quad \forall (i,j) \in \mathscr{O},$$

  gives (informally)

$$\frac{\partial}{\partial t} X(t,\xi) = \Delta H(X(t,\xi) - X^c(\xi)),$$

  where X is the continuous height-density function.
- In addition we impose zero Dirichlet boundary conditions:

$$H(X(t,\xi) - X^c(\xi)) = 0, \quad \text{on } \partial\mathscr{O}.$$

Continuum limit

- We will restrict to the supercritical case, i.e. supposing $x_0 \geq X^c$.
- Substituting $X \to X - X^c$ then yields

$$\frac{\partial}{\partial t}X(t,\xi) = \Delta H(X(t,\xi)),$$
$$X(0,\xi) = x_0(\xi)$$

with $x_0 \geq 0$ and zero Dirichlet boundary conditions:

$$H(X(t,\xi)) = 0, \quad \text{on } \partial\mathscr{O}.$$

- In other words $X^c \equiv 0$ without loss of generality.
- In this case also $X \geq 0$, so that equivalently

$$\frac{\partial}{\partial t}X(t,\xi) = \Delta \mathrm{sgn}(X(t,\xi)),$$
$$X(0,\xi) = x_0(\xi).$$

Continuum limit

- Types of sandpile models:

model	**evolution/threshold depending on**
critical height	height h
critical slope	first derivatives of height (leads to infinity-Laplacian)
Laplacian models	second derivatives of height

- Here we are considering a critical height model

Finite time extinction and SOC

- In [Díaz-Guilera; EPL (Europhysics Letters), 1994], [Giacometti, Diaz-Guilera; Phys. Rev. E, 1998], [Díaz-Guilera; Phys. Rev. A, 1992] it was pointed out that it is more realistic to include stochastic perturbations.
- This leads to SPDE of the form

$$dX_t \in \Delta \mathrm{sgn}(X_t)dt + B(X_t)dW_t,$$

with appropriate diffusion coefficients B.

Finite time extinction and SOC

- Self-organized criticality: We quote from [Bantay, Ianosi; Physica A, 1992]

> *"Criticality" refers to the power-law behavior of the spatial and temporal distributions, characteristic of critical phenomena.*
> *"Self-organized" refers to the fact that these systems naturally evolve into a critical state without any tuning of the external parameters, i.e. the critical state is an attractor of the dynamics.*

- Does SOC appear in the SPDE above?
- Several related and partial results: e.g. [Barbu, 2013], [Röckner, Wang; 2011], [Barbu, Da Prato, Röckner; 2012], [Barbu, Röckner; 2012], [Barbu, Da Prato, Röckner; 2009].
- Very recently (preprint coming soon) we managed to prove finite time extinction, i.e. for

$$\tau_0 = \inf\{t \geq 0 | X_t(\xi) = 0, \text{ for a.e. } \xi \in \mathcal{O}\}$$

we have

$$\mathbb{P}[\tau_0 < \infty] = 1.$$

Thanks

Thanks!