Synchronization for the stochastic quantisation equation

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Introduction: Stochastic quantization and sampling

Euclidean quantum field theory: Formal ϕ^4_d measure

$$\mu \quad " = " \quad \frac{1}{Z} e^{-\int \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 - \frac{1}{2}|u|^2} du$$
$$" = " \quad \frac{1}{Z} e^{-\int \frac{1}{4}|u|^4 - \frac{1}{2}|u|^2} d\nu(u)$$

for ν Gaussian free field on $\mathbb{R}^d$.

Parisi and Wu stochastic quantization: Langevin dynamics for which μ is (formally) invariant

$$\partial_t u = \Delta u - u^3 + u^2 + \sqrt{2}\,\xi$$
$$= -\frac{D}{Du}\Big(\int \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 - \frac{1}{2}|u|^2\Big) + \sqrt{2}\,\xi.$$

Lots of activity (Albeverio–Röckner, Da Prato–Debussche, Hairer–..., R.–Zhu–Zhu, Gubinelli–Hofmanova, ...).

Aims:

- Construct the ϕ^4_d measure as the invariant measure to the quantization equation
- Sample the ϕ^4_d measure via the quantization equation

Introduction: Sampling and synchronization [Lemaire, Pages, Panloup; 2015]

Consider the simpler setting

$$dX_t = b(X_t)dt + dB_t$$

with i.p.m. μ. Want to sample via $\frac{1}{t}\int_0^t \delta_{X_r(\omega)} dr \approx \mu$ for t large.
Euler scheme

$$X_{n+1} = X_n + \gamma_{n+1} b(X_n) + \sqrt{\gamma_{n+1}} U_{n+1}$$

with U_n i.i.d. normals, step-size $\gamma_n \to 0$, $\Gamma_n := \sum_{k=1}^n \gamma_k \to \infty$.Let

$$\mu_n(\omega) := \frac{1}{\Gamma_n} \sum_{k=1}^{n} \gamma_k \delta_{X_{k-1}(\omega)}.$$

Then

$$\mu_n(\omega) \to^* \mu. \qquad \text{(LLN)}$$

Rate of convergence: Let $\Gamma_n^{(2)} := \sum_{k=1}^n \gamma_k^2$. If $\frac{\Gamma_n^{(2)}}{\Gamma_n} \to 0$,then

$$\sqrt{\Gamma_n}(\mu_n(\omega, f) - \mu(f)) \to^* N(0, \int |\nabla g_f|^2 d\mu), \qquad \text{(CLT)}$$

with $f - \mu(f) = Ag_f$.

Polynomial step $\gamma_k = Ck^{-\mu}$. Optimal rate $\mu > 1/3$ and $\sqrt{\Gamma_n} \approx n^{\frac{1}{3}}$.

Take away message: Need that γ_k tends to zero fast enough otherwise discretization error takes over.

This limits the rate of convergence

Idea: Use Richardson–Romberg extrapolation to remove the bias for larger time speps.

Let $\tilde{\gamma}_{2n} = \frac{\gamma_n}{2}$. Euler with half-step size

$$Y_{n+1} = Y_n + \tilde{\gamma}_{n+1} b(Y_n) + \sqrt{\tilde{\gamma}_{n+1}} U_{n+1}.$$

Then (X_n, Y_{2n}) is Euler for the two-point motion (X_t, X_t).

Set

$$\mu_n^2(\omega) := \frac{1}{\Gamma_n} \sum_{k=1}^{n} \gamma_k (\delta_{Y_{2(k-1)}(\omega)} + \delta_{Y_{2k-1}(\omega)} - \delta_{X_k(\omega)}).$$

Let $\Gamma_n^{(3)} := \sum_{k=1}^{n} \gamma_k^3$. If $\frac{\Gamma_n^{(3)}}{\Gamma_n} \to 0$

then

$$\sqrt{\Gamma_n}(\mu_n^2(\omega, f) - \mu(f)) \to^* N(0, \sigma^2), \qquad \text{(CLT)}$$

Allows to increase the step-size $\gamma_k = Ck^{-\mu}$ to $\mu > 1/5$. Gives rate $\sqrt{\Gamma_n} \approx n^{\frac{2}{5}}$.

But: Might increase the variance σ^2.

We have

$$\sigma^2 = 5 \int |\nabla g_f|^2 d\mu - 4 \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla g_f(x) \cdot \nabla g_f(y) \mu^2(dx, dy)$$

where μ^2 is an invariant measure for the two-point motion (X_t, X_t).

Note: σ^2 is minimal for $\mu^2(dx, dy) = \mu_\Delta$. I.e. need to show synchronization to show that variance does not increase by RR-extrapolation.

When does synchronisation by noise occur?

- Consider a (scalar-valued) PDE of the form

$$\begin{cases} \partial_t u = \Delta u + b(u) & (t,x) \in \mathbb{R}_{>0} \times \mathbb{T}^d \\ u|_{t=0} = f & x \in \mathbb{T}^d. \end{cases} \tag{1}$$

  The long time behaviour of u might be chaotic.
- Consider the corresponding SPDE

$$\begin{cases} \partial_t u = \Delta u + b(u) + \sqrt{2}\,\xi & (t,x) \in \mathbb{R}_{>0} \times \mathbb{T}^d \\ u|_{t=0} = f & x \in \mathbb{T}^d \end{cases} \tag{2}$$

  where ξ is a random forcing, e.g. space-time white noise. What about the long time behaviour of the stochastic dynamics?
- We say that synchronisation by noise occurs when the random attractor $\mathcal{A}(\omega)$ of (2) is a singleton $\{a(\omega)\}$ almost surely.
- When synchronisation by noise occurs, trajectories of (2) convergence to each other.

Order-preserving random dynamical systems (RDS): [Fandoli-Gess-Scheutzow '17]: Let φ be order-preserving, strongly mixing with i.p.m. μ. Assume, $\forall \varepsilon > 0$, $\exists [f, g]$ such that

$$\mu([f, g]) \geq 1 - \varepsilon.$$

Then weak synchronization holds.

E.g. ok if $\mu(L^\infty) = 1$.

For $d = 1$: For

$$du = \Delta u - u^3 + u\, dt + \sqrt{2}\,\xi$$

we have

$$|u(t; f) - u(t; g)| \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad t \to \infty \quad \text{for all} \quad f, g \in \mathbb{R}.$$

The random attractor $\mathcal{A}(\omega) = \{a(\omega)\}$ is such that $\mathrm{Law}(a) = \mu$.

Question: Higher dimension?

- If ξ is space-time white noise, even for $b(u) = -u^3 + u$, $(\star)$ is singular for $d \geq 2$. Solution theory via renormalisation applicable in $d = 2, 3$, but u is a Schwartz distribution.
- Order preservation? Natural choice?
- Strong mixing properties hold in $d = 2, 3$.

Back to SPDEs; Motivation/Framework

Order preserving Markov semigroups (Butkovsky–Scheutzow '20):

- For $p \in [1, \infty)$ integer, let $\varphi_p(f) := 2^{p-1} \int \operatorname{sgn}(f)|f|^p dx$.
- $g \leq f \Rightarrow \|f - g\|_p^p \leq \varphi_p(f) - \varphi_p(g)$ (*φ_p non-decreasing).*
- $|\varphi_p(f) - \varphi_p(g)| \lesssim (\|f - g\|_p \wedge 1) \max\{1, \|f\|_p^p + \|g\|_p^p\}$.

If u is a stochastic semi-flow which preserves "$\leq$" and satisfies nice L^p bounds then for $g \leq f$:

$$\mathbb{E}\|u(t; f) - u(t; g)\|_p^p \lesssim_{f,g} \mathcal{W}_{\|\cdot - \cdot\|_p \wedge 1; p}\big(u(t; f), u(t; g)\big).$$

Remark: In particular, mixing with respect to $\mathcal{W}_{\|\cdot - \cdot\|_p \wedge 1; p}$ implies convergence for trajectories of ordered initial data to each other.

Double-well in infinite dimensions

We want to study synchronisation for

$$\begin{cases} \partial_t u = \Delta u - u^3 - 3\infty u + u + \sqrt{2}\,\xi & (t,x) \in \mathbb{R}_{>0} \times \mathbb{T}^d \\ u|_{t=0} = f & x \in \mathbb{T}^d. \end{cases} \tag{3}$$

Remark

- Eq. (3) needs to be renormalised in $d = 2, 3$.
- Solutions are not functions, but Schwartz distributions.
- The Butkovsky–Scheutzow framework is not directly applicable.
- We need a partial order on the space of Schwartz distributions *compatible with a norm*.

Generalisation of the Butkovsky–Scheutzow framework

Define the partial order "$\preceq$" on the space of Schwartz distributions:

$$g \preceq f \text{ iff } \langle g, \varphi\rangle \leq \langle f, \varphi\rangle \text{ for every non-negative } \varphi \in \mathcal{C}^\infty.$$

Observation: There is a natural generalisation of the Butkovsky–Scheutzow framework to negative Besov norms compatible with the partial order "$\preceq$":

$$\|f\|_{-\alpha;p}^p := \int_0^1 s^{\frac{\alpha p}{2}} \|e^{s\Delta} f\|_p^p \frac{ds}{s}, \quad \varphi_{-\alpha;p}(f) := \int_0^1 s^{\frac{\alpha p}{2}} \varphi_p(e^{s\Delta} f) \frac{ds}{s},$$

where $\alpha \in (0,1)$.

It is easy to see that the same properties hold:

- $g \preceq f \Rightarrow \|f-g\|_{-\alpha;p}^p \leq \varphi_{-\alpha;p}(f) - \varphi_{-\alpha;p}(g)$.
- $|\varphi_{-\alpha;p}(f) - \varphi_{-\alpha;p}(g)| \lesssim (\|f-g\|_{-\alpha;p} \wedge 1) \max\{1, \|f\|_{-\alpha;p}^p + \|g\|_{-\alpha;p}^p\}$.

Corollary: If u preserves "$\preceq$", satisfies good $\|\cdot\|_{-\alpha;p}$-bounds and mixing with respect to $\mathcal{W}_{\|\cdot - \cdot\|_{-\alpha;p} \wedge 1;p}$ holds, trajectories of ordered initial data converge to each other.

Extend to arbitrary f, g; Extra ingredient: "Coming down from infinity" (T.–Weber, Mourrat–W.)

For all $t > 0$, there exist a random constant $K > 0$, with $\mathbb{E}K^q < \infty$ for every $q \geq 1$, and exponents $0 < \alpha < \alpha_0$ such that

$$\sup_{\|f\|_{-\alpha_0;\infty} < \infty} \|u(t; f)\|_{-\alpha;\infty} \leq K.$$

- Here, it allows us to construct trajectories $u_\pm(t)$ such that for every f
$$u_-(t) \preceq u(t; f) \preceq u_+(t).$$

- Uniform in the initial data exponential mixing in total variation implies
$$\mathcal{W}_{\|\cdot - \cdot\|_{-\alpha;p} \wedge 1; p}\big(u_+(t), u_-(t)\big) \lesssim \mathrm{e}^{-\lambda_* t}.$$

Uniform synchronisation by noise...

Theorem (Gess, Tsatsoulis 19; Stochastic quantisation in $d = 2, 3$)
For $p \in [1, \infty)$ quantified uniform synchronisation by noise in the initial data occurs:

$$\mathbb{E}^{\frac{1}{p}} \left(\sup_{f,g} \|u(t; f) - u(t; g)\|_{-\alpha-\frac{d}{p};\infty} \right)^{p} \lesssim \mathrm{e}^{-\frac{\lambda_*}{p} t}.$$

Furthermore, $\mathcal{A}(\omega) = \{a(\omega)\}$ where $\mathrm{Law}(a) = \mu$.

References

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