Stochastic scalar conservation laws

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joint work with: Panagiotis E. Souganidis
[G., Souganidis; *CMS*, 2014], [G., Souganidis; *arXiv*, 2015].

Outline

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Motivation

Motivation

Motivation

- We will consider PDE driven by a 'rough' signal z of the type

$$du + \operatorname{div}(A(x,u) \circ dz) = 0.$$

If A is a diagonal matrix this becomes

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \circ dz_j = 0$$

- The motivation comes from two directions: Relation to Hamilton-Jacobi equations, mean-field games.

Motivation

- In the one-dimensional case: If v solves the Hamilton-Jacobi equation

$$dv + A(\partial_x v, x) \circ dz = 0$$

then $u = \partial_x v$ solves

$$du + \partial_x A(v, x) \circ dz = 0.$$

- But: The mathematical methods available for Hamilton-Jacobi equations (viscosity solutions) and scalar conservation laws (entropy solutions, kinetic methods) are very different.

Motivation

- Mean-field games going back to Lasry, Lions: Consider the SDE

$$dX_t^i = \sigma\left(X_t^i, \frac{1}{L-1}\sum_{j\neq i}\delta_{X_t^j}\right)\circ dz_t \quad \text{in } \mathbb{R}^N$$

  for $i = 1, \ldots, L$.
- Then the empirical law of X converges to a measure π_t with density m_t which evolves according to

$$dm + \text{div}(\sigma^*(x,m)\circ dz) = 0.$$

- Note that in general σ^* is not a diagonal matrix. We need the full generality of

$$du + \text{div}(A(x,u)\circ dz) = 0.$$

Spatially homogeneous case

Well-posedness - Spatially homogeneous case

Spatially homogeneous case

- We start with the spatially homogeneous case and A being a diagonal matrix, that is:

$$du + \sum_j \partial_{x_j} A(u) \circ dz_j = 0. \tag{SSCL}$$

  Here z is assumed to be a continuous function ('rough' = continuous).
- If z is smooth, then (SSCL) makes sense classically

$$du + \sum_j \partial_{x_j} A_j(u) \dot{z}_j = 0.$$

- Aims:
  - Intrinsic solution: Define solutions to (SSCL) and prove well-posedness.
  - Consistency: Show that solutions to (SSCL) are obtained by approximation of the driving signal z.

Spatially homogeneous case

Reminder:

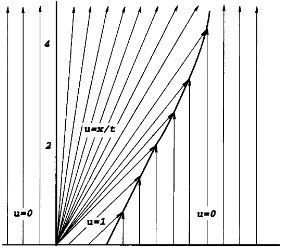
- Solutions to (deterministic) scalar conservation laws

$$du + \sum_j \partial_{x_j} A_j(u)\dot{z}_j = 0$$

develop shocks (discontinuities)

- e.g. Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = 1_{[0,1]}$$



- At shocks, weak solutions are non-unique.
- Physically right solution is selected by entropy inequalities [Kružkov, 1970]

$$dS(u) + \sum_j \partial_{x_j} Q_j(u)\dot{z}_j \le 0.$$

Spatially homogeneous case

- Alternative: kinetic solutions [Lions, Perthame, Tadmor; *JAMS*, 1994].
- For simplicity say $u_0 \geq 0$, which implies $u \geq 0$.
- We consider the characteristic function

$$\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi).$$

Elementary calculation (if u were smooth, i.e. no shocks):

$$\begin{aligned}
\partial_t \chi(t,x,\xi) :&= \delta_{\xi=u(t,x)} \partial_t u(t,x) = -\delta_{\xi=u(t,x)} \sum_j \partial_{x_j} A_j(u) \dot{z}_j \\
&= -\delta_{\xi=u(t,x)} \sum_j A_j'(u) \partial_{x_j} u \dot{z}_j = -\delta_{\xi=u(t,x)} \sum_j A_j'(\xi) \partial_{x_j} u \dot{z}_j \\
&= -\sum_j A_j'(\xi) \partial_{x_j} 1_{[0,u(t,x)]}(\xi) \dot{z}_j = -\sum_j A_j'(\xi) \partial_{x_j} \chi(t,x,\xi) \dot{z}_j.
\end{aligned}$$

Spatially homogeneous case

- This is true up to shocks. The shocks introduce an error, the '*entropy dissipation measure*' m:

$$\partial_t \chi(t,x,\xi) + \sum_j A_j'(\xi)\partial_{x_j}\chi(t,x,\xi)\dot{z}_j = \partial_\xi m. \tag{1}$$

- In deterministic setting: u is an entropy solution iff $\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi)$ is a kinetic solution to (1).
- Advantage: (1) is a linear equation in χ, at the expense of introducing the additional velocity variable ξ.
- In contrast to the non-linear situation, (1) can be transformed in a 'robust' form, i.e. in a form making sense also for non-smooth z.
- Here we follow the principle idea of stochastic viscosity solutions, i.e. do not transform the PDE itself, but put the transformation into test-functions.

Spatially homogeneous case

- Choose the test-functions φ as solutions to the transport equation

$$\partial_t \varphi(t,x,\xi) + \sum_j A_j'(\xi)\partial_{x_j}\varphi(t,x,\xi)\dot{z}_j = 0. \tag{2}$$

- Then consider *convolutions along characteristics:*

$$\begin{aligned}
\partial_t \chi * \varphi &= \partial_t \int \chi(t,x,\xi)\varphi(t,x,\xi)dxd\xi \\
&= \int \left(-\sum_j A_j'(\xi)\partial_{x_j}\chi(t,x,\xi)\dot{z}_j + \partial_\xi m \right) \varphi(t,x,\xi)dxd\xi \qquad (3) \\
&+ \int \chi(t,x,\xi) \left(\sum_j A_j'(\xi)\partial_{x_j}\varphi(t,x,\xi)\dot{z}_j \right) dxd\xi \\
&= \int \partial_\xi m \varphi(t,x,\xi)dxd\xi.
\end{aligned}$$

- The point is that φ in (2) is well-defined also for continuous z, thus (3) is well-defined for z continuous
  $\rightarrow$ use (3) as the a definition of a solution: *pathwise entropy solution.*

Spatially homogeneous case

- It remains to give meaning to

$$\partial_t \varphi(t,x,\xi) + \sum_j A_j'(\xi)\partial_{x_j}\varphi(t,x,\xi)\dot{z}_j = 0 \tag{4}$$

for continuous signals z.
- Method of characteristics for (4) gives

$$\varphi(t,x,\xi) = \varphi^0(x + A'(\xi)z_t).$$

Theorem (Lions, Perthame, Souganidis; SPDE 2013)

1. *For each $u_0 \in L^\infty \cap BV$ there is a pathwise entropy solution.*
2. *Let $u^{(1)}, u^{(2)} \in L^\infty([0,T]; BV(\mathbb{R}^N))$ be two pathwise entropy solutions with driving signals $z^{(1)}, z^{(2)} \in C_0([0,T];\mathbb{R}^N)$. Then,*

$$\|u^{(1)}(t) - u^{(2)}(t)\|_{L^1(\mathbb{R}^N)} \le \|u_0^1 - u_0^2\|_{L^1(\mathbb{R}^N)} + C\sqrt{\|z^{(1)} - z^{(2)}\|_{C([0,t];\mathbb{R}^N)}}.$$

- In particular, this yields consistency.

Spatially homogeneous case

Comments on the proof:

- Want to estimate

$$\partial_t \int |u^1 - u^2| dy = \partial_t \int |\chi^1 - \chi^2|^2 dy d\xi .$$

- To use definition estimate instead

$$\partial_t \int |\chi^1 * \varphi^\varepsilon - \chi^2 * \varphi^\varepsilon|^2 dy d\xi$$

  with φ^ε test-functions transported along characteristics.
- *Doubling the variables*: One considers a family of testfunctions

$$\varphi^\varepsilon(t,x,y,\xi) = \varphi^{0,\varepsilon}(x - y + A'(\xi)z_t) \xrightarrow[\varepsilon \to 0]{} \delta(x - y + A'(\xi)z_t).$$

Spatially homogeneous case

- Leads to error terms, due to doubling of the variables, that need to be controled. An important one:

$$\begin{aligned}
&\partial_\xi \varphi^\varepsilon(x,y,\xi,t)\varphi^\varepsilon(x,y',\xi,t)+\varphi^\varepsilon(x,y,\xi,t)\partial_\xi \varphi^\varepsilon(x,y',\xi,t)\\
&=(D\varphi^\varepsilon)(x,y,\xi,t)A''(\xi)z_t\varphi^\varepsilon(x,y',\xi,t)+\varphi^\varepsilon(x,y,\xi,t)(D\varphi^\varepsilon)(x,y',\xi,t)A''(\xi)z_t\\
&=\partial_y \varphi^\varepsilon(x,y,\xi,t)A''(\xi)z_t\varphi^\varepsilon(x,y',\xi,t)+\varphi^\varepsilon(x,y,\xi,t)\partial_y \varphi^\varepsilon(x,y',\xi,t)A''(\xi)z_t\\
&=\partial_y(\varphi^\varepsilon(x,y,\xi,t)\varphi^\varepsilon(x,y',\xi,t))A''(\xi)z_t,
\end{aligned}$$

  which vanishes after an integration in y
  $\rightarrow$ crucial cancellation which uses the simple structure of the characteristics.

Spatially inhomogeneous case

Well-posedness - Spatially inhomogeneous case

Spatially inhomogeneous case

- Let us now consider the spatially inhomogeneous case:

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \circ dz_j = 0.$$

- The principle idea remains the same: We pass to the kinetic formulation:

$$d\chi + \sum_{j=1}^{N} (\partial_u A_j)(x,\xi) \partial_{x_j} \chi \dot{z}_j + \left(\sum_{j=1}^{N} \operatorname{div}_{x_j} A(x,\xi) \dot{z}_j \right) \partial_\xi \chi = \partial_\xi m.$$

- Again we test by solutions to

$$\partial_t \varphi(t,x,\xi) + \sum_{j=1}^{N} (\partial_u A_j)(x,\xi) \partial_{x_j} \varphi \dot{z}_j + \left(\sum_{j=1}^{N} \operatorname{div}_{x_j} A(x,\xi) \dot{z}_j \right) \partial_\xi \varphi = 0.$$

- As before one gets: *convolution along characteristics*

$$\partial_t \chi * \varphi = \int \partial_\xi m \varphi(t,x,\xi) dx d\xi.$$

Spatially inhomogeneous case

- The difficulty lies in the characteristics to

$$\partial_t \varphi(t,x,\xi) + \sum_{j=1}^{N} (\partial_u A_j)(x,\xi)\partial_{x_j}\varphi \dot{z}_j + \left(\sum_{j=1}^{N} \mathrm{div}_{x_j} A(x,\xi)\dot{z}_j \right) \partial_\xi \varphi = 0.$$

- In contrast to the spatially homogeneous case, the characteristics do not have an explicit solution anymore. Instead they are given as the solutions to the rough DE:

$$dX^i_{(t_1,x,\xi)}(t) = \sum_{j=1}^{M} (\partial_u A_j)(X_{(t_1,x,\xi)}(t), \Xi_{(t_1,x,\xi)}(t))dz^{t_1,j}(t),$$

$$d\Xi_{(t_1,x,\xi)}(t) = -\sum_{j=1}^{M} (\partial_{x_j} A_j)(X_{(t_1,x,\xi)}(t), \Xi_{(t_1,x,\xi)}(t))dz^{t_1,j}(t),$$

$$X^i_{(t_1,x,\xi)}(0) = x^i \text{ and } \Xi_{(t_1,x,\xi)}(0) = \xi.$$

- We get

$$\varphi_{t_0}(x,\xi,t) = \varphi^0 \begin{pmatrix} X_{(t,x,\xi)}(t-t_0) \\ \Xi_{(t,x,\xi)}(t-t_0) \end{pmatrix}.$$

Spatially inhomogeneous case

- Hence, to get well-posedness of φ we need stability of (X,Ξ) with respect to the driving signal. I.e. rough path stability.
  $\rightarrow$ need z to be a rough path.

Theorem (Gess, Souganidis; CMS, 2015)

1. *For each $u_0 \in L^1 \cap L^2$ there is a pathwise entropy solution.*
2. *Let $u^{(1)}, u^{(2)} \in L^\infty([0,T]; L^1(\mathbb{R}^N))$ be two pathwise entropy solutions with the same driving signal z. Then,*

$$\|u^{(1)}(t) - u^{(2)}(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0^1 - u_0^2\|_{L^1(\mathbb{R}^N)}.$$

Spatially inhomogeneous case

Comments on the proof:

- One looses the cancellation effect from the homogeneous case.
- Instead, the error has to be carefully controled.
- Key new step: Interval splitting + rough path estimates for the characteristics
- Drawback: Do not get a quantitative continuous dependence on the driving rough paths anymore.

Long-time behavior

Long-time behavior

Long-time behavior

- We aim to analyze the long-time behavior of

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta_j = 0$$

  on the torus $\mathbb{T}^N$.
- We will show

$$u(t) \to \bar{u}_0 = \int_{\mathbb{T}^N} u_0(x)dx \quad \text{for } t \to \infty$$

  in $L^1(\mathbb{T}^N)$.

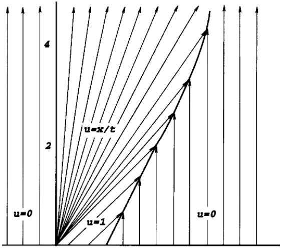
Some results from the deterministic case

- We first recall some (known) results from the deterministic case, i.e. for

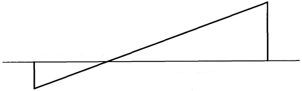
$$\partial_t u + \sum_{j=1}^{N} \partial_{x_j} A_j(u) = 0.$$

- Recall: Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = 1_{[0,1]}$$



- For large times get asymptotic shape: "N-wave"



Some results from the deterministic case

Some existing results:

- $N=1$: [Lax; *CPAM*, 1957]. Rate for strictly convex flux:
$$\|u(t)-\bar{u}_0\|_\infty \lesssim t^{-1}.$$
- $N=2$: [Engquist, E; *CPAM*; 1993]. Rate for strictly convex flux:
$$\|u(t)-\bar{u}_0\|_\infty \lesssim t^{-1}.$$
- $N\geq 1$: [Chen, Frid; *ARMA*; 1999], [Chen, Perthame; *Proc. AMS*; 2009]. If A is 'genuinely nonlinear' then
$$u(t)\to \bar{u}_0 := \int_{\mathbb{T}^N} u_0\,dx \quad \text{for } t\to\infty,$$
in $L^1(\mathbb{T}^N)$.
- For general $N\geq 3$ no rate of convergence known!

Some results from the deterministic case

Idea of the proof for general $N \geq 1$:

- Consider pullback limit, i.e. let $u(s,t,u_0)$ be the solution started in u_0 at time $s \leq 0$.
- Key point: By averaging Lemma (genuine nonlinearity of A), the solution operator $S_t : L^\infty \to L^1$ is locally compact. Thus one can extract a subsequence such that

$$u(s,t,u_0) \to v(t)$$

  for $s \to -\infty$.
- The limit v is a solution to

$$\partial_t v + \sum_{j=1}^{N} \partial_{x_j} A_j(v) = 0$$

  for all time $t \in \mathbb{R}$.
- But such a function has to be constant (again via averaging techniques).

(New) rates for the deterministic case

- Assume that the flux A is genuinely nonlinear, in the sense that: there exist $\theta \in (0,1]$ and $C > 0$ such that, for all $\sigma \in S^{N-1}$, $z \in \mathbb{R}$ and $\varepsilon > 0$,

$$|\{\xi \in \mathbb{R} : |A'(\xi) \cdot \sigma - z| \leq \varepsilon\}| \leq C\varepsilon^{\theta}.$$

- For example: For A strictly convex, $N = 1$ we have $\theta = 1$.
- Let u be the unique entropy solution to

$$\partial_t u + \sum_{j=1}^{N} \partial_{x_j} A_j(u) = 0.$$

Theorem (G., Souganidis; 2015)

For $t \geq 1$ and $u_0 \in L^2(\mathbb{T}^N)$,

$$\|u(\cdot,t;\cdot,u_0) - \bar{u}_0\|_1 \leq t^{-\frac{\theta}{2+\theta}} \left(\|u_0\|_2^2 + 1\right).$$

(New) rates for the stochastic case

- Let us return to

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta_j = 0.$$

- Again assume that A is genuinely nonlinear.

Theorem (G., Souganidis; 2015)

For $t \geq 1$ and $u_0 \in L^2(\mathbb{T}^N)$,

$$\mathbb{E}\|u(\cdot,t;\cdot,u_0) - \bar{u}_0\|_1 \leq \left(\|u_0\|_2^2 + 1\right) t^{-\frac{\theta}{3+\theta}}.$$

- E.g. $\theta = 1$: deterministic rate $t^{-\frac{1}{3}}$, stochastic rate $t^{-\frac{1}{4}}$. But: No claim of optimality.
- Note: Brownian motion scales like $\sqrt{t}$, which "slows down" characteristics.

Regularization by noise

Regularization by noise

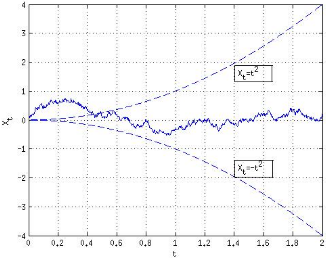
Regularization by noise - linear case

- We recall: Consider
$$du + b(x)\cdot\nabla u = 0 \tag{TE}$$
for non-Lipschitz b (but, say, Hölder continuous). E.g. $b(x) = \mathrm{sgn}(x)\sqrt{|x|}$.
- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than $u(t) \in BV$ even if u_0 is smooth.
- In contrast: Consider
$$du + b(x)\cdot\nabla u = -\nabla u \circ dW_t. \tag{STE}$$
- Then, e.g. [Flandoli, Fedrizzi; *JFA*, 2013]: If u_0 is smooth then $u(t)$ is smooth, i.e. regularization by noise.

Regularization by noise - linear case

- Consequence: Well-posedness by noise, i.e. (STE) is well-posed in cases for which (TE) is not [Flandoli, Gubinelli, Priola; *Invent. Math.*, 2010]. E.g.

$$du + b(x) \cdot \nabla u = -\nabla u \circ dW_t$$
$$b(x) = \mathrm{sgn}(x)\sqrt{|x|}$$



- Entirely open: What about the nonlinear case, e.g. Burgers?

Quasi-solutions and averaging

- Consider the Burgers' equation

$$\partial_t u + \partial_x u^2 = 0 \quad \text{on } \mathbb{T} \tag{B}$$

- Kinetic solution: For some non-negative measure m

$$\partial_t \chi + \xi \partial_x \chi = \partial_\xi m.$$

- By averaging techniques [Jabin, Perthame; 2002], $t > 0$,

$$u(t) \in W^{1,\lambda} \quad \text{for all } \lambda \in (0, \frac{1}{3}).$$

But: Bounded entropy solutions to (B) satisfy $u(t) \in BV$, $t > 0$. Thus, regularity by averaging is not sharp.

Quasi-solutions and averaging

- [De Lellis, Westdickenberg; *AIHP*, 2003]: Say that u is a quasi-solution, if for some Radon measure m

  $$\partial_t \chi + \xi \partial_x \chi = \partial_\xi m.$$

  Then, there is a quasi-solution to (B) such that

  $$u(t) \notin W^{1,\lambda} \quad \text{for all } \lambda > \frac{1}{3},$$

  i.e. regularity by averaging is sharp for quasi-solutions.

(New) results for the stochastic case

- Consider the stochastic Burgers' equation

$$du + \partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T} \tag{SB}$$

Theorem (G., Souganidis; 2015)

Let u be a pathwise quasi-solution to (SB). Then, $t > 0$,

$$u(t) \in W^{\lambda,1} \quad \textit{for all } \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.}.$$

- Thus: quasi-solutions to (SB) are more regular than to (B), i.e. regularization by noise.

Thanks

Thanks!