Path-by-path regularization by noise for scalar conservation laws

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University of Warwick, January 2018

joint work with: Panagiotis E. Souganidis, Benoit Perthame, Paul Gassiat, Mario Maurelli, Khalil Chouk
[G., Souganidis; CMS, 2014], [G., Souganidis; CPAM, 2016],
[G., Perthame, Souganidis; SINUM, 2016], [Gassiat, G.; arXiv:1609.07074],
[Maurelli, G.; arXiv:1701.05393], [Chouk, G.; arXiv:1708.00823].

Outline

- Classical well-posedness for ODE:

$$dX_t^x = b(X_t^x)dt, \quad X_0^x = x$$

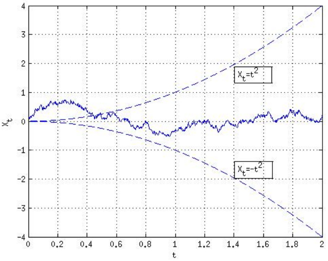
  is well-posed if b is sufficiently smooth, e.g. Lipschitz continuous.

- In contrast, well-posedness for SDE: $(\sigma > 0)$

$$dX_t^x = b(X_t^x)dt + \sigma d\beta_t, \quad X_0^x = x$$

  has a unique solution if b is bounded, measurable. This is called '*well-posedness by noise*'.

- A simple example: $b(x) = 2\mathrm{sgn}(x)\sqrt{|x|}$:



Introduction

- Key hope in SPDE: Establish similar effects for PDE, in particular in fluid dynamics, e.g. $3d$-Navier-Stokes equations, gas dynamics.
- Problem: A-priori unclear which form of noise to consider
- Additive noise, e.g.

$$du = \Delta u\, dt + f(u)\, dt + dW_t \quad \text{[Gyöngy, Pardoux; 1993]}$$

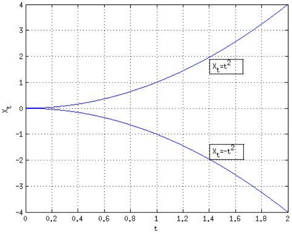
$$du + (u \cdot \nabla) u\, dt + \nabla p\, dt = \Delta u\, dt + dW_t \quad \text{[Flandoli, Romito; 2007].}$$

- More recent: Linear multiplicative noise

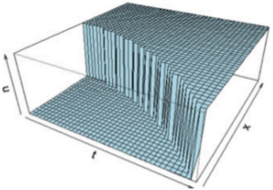
$$du + b(x) \cdot \nabla u\, dt = \nabla u \circ d\beta_t. \quad \text{[Flandoli, Gubinelli, Priola; 2010].}$$

- We recall: Consider
$$\partial_t u + b(x)\cdot\nabla u = 0, \qquad \text{(TE)}$$
for non-Lipschitz b (but, say, Hölder continuous). E.g. $b(x) = 2\mathrm{sgn}(x)\sqrt{|x|}$.
- Characteristics for (TE):
$$dX_t^x = b(X_t^x)dt \in \mathbb{R}^d.$$
- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than $u(t) \in BV$ even if u_0 is smooth.
- Characteristics branch causing non-uniqueness of weak solutions.

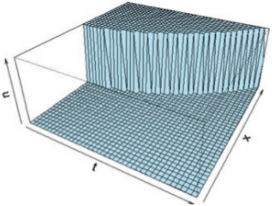
characteristics



weak solutions, $u_0 = 1_{[0,\infty)}$



The solution $u^{(-)}(x,t) = 1_{\{x>-t^2\}}$



The solution $u^{(+)}(x,t) = 1_{\{x\geq t^2\}}$

- Question: Can noise restore uniqueness or increase regularity?

- Consider, $\sigma > 0$,

$$du + b(x)\cdot\nabla u = \sigma\nabla u \circ d\beta_t. \tag{STE}$$

- Characteristics for (STE):

$$dX_t^x = b(X_t^x)dt - \sigma d\beta_t \in \mathbb{R}^d.$$

- Two (related) effects: regularization by noise, well-posedness by noise.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; 2010]: Weak solutions to (STE) are unique.
- Regularization by noise [Flandoli, Fedrizzi; 2013]: If $u_0 \in \bigcap_{p\geq 1} W^{1,p}$ then $u(t) \in \bigcap_{p\geq 1} W^{1,p}$, $\mathbb{P}$-a.s..

- Left open: What about the nonlinear case, e.g. Burgers?
- Linear multiplicative noise does not help anymore:

$$du + \partial_x u^2 \, dt = \partial_x u \circ d\beta_t.$$

Then: $v(t,x) := u(t, x - \beta_t)$ is a solution to

$$\partial_t v + \partial_x v^2 = 0.$$

- In particular, weak solutions are non-unique.
- Conclusion in [Flandoli, Gubinelli, Priola; *Invent. Math.*, 2010]:
  „It is very easy to produce examples [...] for a stochastic version of Euler equation which show that the particular noise we use does not have any regularizing effect in this case. [...] The generalization to nonlinear transport equations, where b depends on u itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem.“
- Different forms of noise?

Regularization by noise for nonlinear SPDE

Regularization by noise in nonlinear SPDE

Regularity of solutions for stochastic SCL

- Consider mean field equations

$$dX_t^i = \sigma^L\left(X_t^i, \frac{1}{L}\sum_{j=1}^{L}\delta_{X_t^j}\right)\circ d\beta_t \quad \text{in } \mathbb{R}^N$$

Taking $L \to \infty$ and $\sigma^L \to \sigma$ leads to stochastic scalar conservation laws

$$du + \operatorname{div}(\underbrace{\sigma(x,u)u}_{=:A(x,u)}\circ d\beta) = 0 \quad \text{on } (0,T)\times\mathbb{R}^d.$$

- Methods apply to general spatially homogeneous and truly nonlinear flux A.
- For simplicity, in this talk restrict to

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0.$$

Consider

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0, \quad \text{on } (0,T)\times\mathbb{R}^d$$
$$u(0) = u_0 \in L^\infty(\mathbb{R}^d).$$

For

$$\chi(t,x,v) = \chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}$$

we get the kinetic form

$$\partial_t \chi + v\partial_x \chi = \partial_v m \quad \text{on } (0,T)\times\mathbb{R}^d\times\mathbb{R}.$$

Dissipation-dispersion approximations lead to

Definition (De Lellis, Otto, Westdickenberg, 2003)

A function $u \in L^\infty([0,T]\times\mathbb{R}^d)$ is said to be a quasi-solution if $\chi(t,x,v) = \chi(u(t,x),v)$ satisfies

$$\partial_t \chi + v\partial_x \chi = \partial_v m \quad \text{on } (0,T)\times\mathbb{R}^d\times\mathbb{R}$$

for some finite (signed) measure m.

Theorem (De Lellis, Westdickenberg, 2003; Jabin, Perthame 2002)

Consider

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0, \quad \text{on } (0,T)\times\mathbb{R}.$$

Then

1. *Each quasi-solution satisfies, for all* $\lambda \in (0, \frac{1}{3})$,

$$u \in L^1([0,T]; W^{\lambda,1}(\mathbb{R})).$$

2. *For each* $\lambda > \frac{1}{3}$ *there exists a quasi-solution* u*, such that*

$$u \notin L^1([0,T]; W^{\lambda,1}(\mathbb{R})).$$

Theorem (G., Souganidis; CPAM, 2016)

Let $u \in L^\infty$ be a quasi-solution to

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T}.$$

Then,

$$u \in L^1([0,T]; W^{\lambda,1}(\mathbb{R})) \quad \text{for all } \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.}.$$

If u is an entropy solution, then

$$u(t) \in W^{\lambda,1}(\mathbb{R}) \quad \text{for all } t > 0, \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.}. \tag{$\star$}$$

Two resulting questions:

1. Can the zero set in $(\star)$ be chosen uniformly in t?
2. Characterize the properties of Brownian paths leading to $(\star)$.

Regularization by nonlinear noise

- Consider, for $w \in C([0,T])$,

$$du + \frac{1}{2}\partial_x u^2 \circ dw_t = 0, \quad \text{on } \mathbb{R}.$$

- Get

$$\|u(t)\|_{W^{1,\infty}_x} \leq \Big(\max_{0\leq s\leq t}(w(s)-w(t)) \wedge (w(t) - \min_{0\leq s\leq t} w(s))\Big)^{-1}.$$

- Decisive path property: "Changing sign of the derivative".
- For $w = \beta$ we get

$$v(t) \in W^{1,\infty}(\mathbb{R}), \quad \mathbb{P}-a.s.$$

- But: Zero set depends on time $t > 0$.

Path-by-path regularization by noise

Path-by-path regularization by noise

Framework

- Model example:

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{T},$$

  with $w \in C([0,T];\mathbb{R})$.
- Again: Results are given for general truly nonlinear flux A.
- How to classify pathwise properties of w leading to improved regularity?

$$\begin{tikzcd}
& w = \beta(\omega) \text{ satisfies pathwise property} \arrow[dr, "\text{[Chouk, G.]}"] & \\
\beta \text{ Brownian motion} \arrow[ur, "\mathbb{P}\text{-f.s.}"] \arrow[rr, "\mathbb{P}\text{-f.s.}", "\text{[G., Souganidis]}"'] & \arrow[u, phantom, "\circlearrowleft"] & u(t,\omega) \in W^{\lambda,1}
\end{tikzcd}$$

Idea of the proof

- Ideas of the proof of regularity for

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T}.$$

- Kinetic formulation:

$$d\chi + v\partial_x\chi \circ d\beta_t = \partial_v m,$$

  for some finite Radon measure m.
- Change of variables gives

$$\chi(t,x,v) = \chi_0(x+v\beta_t, v) + \int_0^t \partial_v m(s, x+v(\beta_t-\beta_s), v)ds.$$

Idea of the proof

- Averaging over velocity

$$u(t,x) = \int_v \chi = \int_v \chi_0(x + v\beta_t, v)dv + \int_0^t \int_v \partial_v m(s, x + v(\beta_t - \beta_s), v)dvds.$$

- The averaging effect appears since the velocity average in v contains averaging of the x-variable.
- Rigorously, this can be seen by Fourier transform, that is,

$$\hat{u}(t,n) = \int_v e^{-iv\beta_t n}\hat{\chi}_0(n,v)dv + \int_0^t \int_v e^{-iv(\beta_t - \beta_s)n}\partial_v \hat{m}(s,n,v)dvds.$$

- The oscillatory integrals have a regularizing effect, both in v and in $\beta_t - \beta_s$.

Framework

- For SDE this has been considered by [Catellier, Gubinelli; *SPA*, 2016]: A path $w \in C(\mathbb{R}_+;\mathbb{R}^d)$ is said to be (ρ,γ)-irregular if

$$|\int_s^t e^{in\cdot w_r}dr| \lesssim (1+|n|)^{-\rho}|t-s|^{\gamma} \quad \forall n \in \mathbb{R}^d, s<t.$$

- Note:

$$\int_s^t e^{in\cdot w_r}dr = \int_{\mathbb{R}} e^{in\cdot x}dL_w^{s,t}(x) = \hat{L_w^{s,t}}(n)$$

the Fourier transform of the local time.

Main result

Theorem

Let $w \in C^{\eta}([0,T],\mathbb{R}^d)$ for some $\eta > 0$ be (ρ,γ)-irregular, u a bounded quasi-solution solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad on\ \mathbb{T}.$$

Then, for all

$$\lambda < \frac{\rho(\eta+1)-(1-\gamma)}{(\rho \vee 1)(\eta+1)+(1-\gamma)},$$

we have

$$\|u\|_{L^1([0,T];W^{\lambda,1}(\mathbb{R}))} < \infty.$$

Corollary

Let β^H be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}]$ and u be a bounded quasi-solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta_t^H = 0 \quad \text{on } \mathbb{T}. \tag{1}$$

Then, for all $\lambda < \frac{1}{1+2H}$,

$$\|u\|_{L^1([0,T];W^{\lambda,1}(\mathbb{R}))} < \infty.$$

- Note: Fully recover the probabilistic result from [G., Souganidis; *CPAM*, 2016]: For $H = \frac{1}{2}$ get $\lambda < \frac{1}{2}$.

A path-by-path scaling condition

A path-by-path scaling condition

Discussion of the path classification

- The proof given in [G., Souganidis; *CPAM*, 2016] uses the *scaling property* of Brownian motion and independence of increments.
- However: (ρ,γ)-irregularity depends on two parameters, also encoding a time regularity. Hence, does not seem to be optimal.
- Moreover: (ρ,γ)-irregularity not easy to check.
- To avoid the use of oscillatory integrals: Completely avoid Fourier methods in the proof.

Idea of the proof

- Consider
$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0.$$
- Kinetic form
$$\partial_t \chi(t,x,v) + v\partial_x \chi(t,x,v) \circ dw_t = \partial_v m(t,x,v).$$
- Rewrite as, for $\lambda > 0$,
$$\partial_t \chi(t,x,v) + v\partial_x \chi(t,x,v) \circ dw_t + \lambda \chi(t,x,v) = \partial_v m(t,x,v) + \lambda \chi(t,x,v).$$

Idea of the proof

- Change of variables

$$\chi(t,x,v) = e^{-\lambda t}\chi(0,x-vw_{0,t},v) + \int_0^t e^{-\lambda(t-s)}(\partial_v m)(s,x-vw_{s,t},v)\,ds$$
$$+\lambda \int_0^t e^{-\lambda(t-s)}\chi(s,x-vw_{s,t},v)\,ds.$$

- Introduce the random X-ray transform

$$(Tg)(t,x) := \int_0^t \int_v g(s,x-vw_{s,t},v)e^{-\lambda(t-s)}\,dvds$$

- Hence,

$$u := \int_v \chi = T(\partial_v m) + \lambda T\chi.$$

where m is a finite measure and $\chi(t,x,v) := 1_{[0,u(t,x)]}(v)$.

- Strategy: Estimate the regularity of $T(\partial_v m)$, $T\chi$ then use real interpolation.

Path-by-path scaling condition

- This leads to: *Path-by-path scaling condition*: Assume that there is a $\iota \in [\frac{1}{2}, 1]$ such that for every $\sigma \in [0,1)$, $\lambda \geq 1$ we have

$$\int_0^T dr \int_0^{T-r} dt\, e^{-\lambda t} |\underbrace{w_{t+r} - w_r}_{=: w_{r,r+t}}|^{-\sigma} \lesssim \lambda^{-1+\iota\sigma}.$$

- Easy to see: (ρ, γ)-irregularity implies path-by-path scaling.

Theorem

Let u be a bounded quasi-solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \textit{on } \mathbb{R}$$

and suppose that $w \in C^\eta$ satisfies path-by-path scaling. Then, for all $\lambda < \frac{1+\eta-\iota}{1+\eta+\iota}$,

$$\|u\|_{L^1([0,T];W^{\lambda,1}(\mathbb{R}))} < \infty.$$

Thanks

Thanks!