Optimal regularity for the (nonlocal) anisotropic porous medium equation

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based on [G.; JEMS 2021], joint with Jonas Sauer, Eitan Tadmor [G., Sauer, Tadmor; Analysis & PDE, 2020], and joint with Jonas Sauer [G., Sauer; arxiv, 2023]

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Consider nonlocal, anisotropic degenerate PDE

$$\partial_t u = \mathcal{L}(u) \text{ on } (0,T) \times \mathbb{R}^d$$

with $u(0) = u_0 \in L^1(\mathbb{R}^d)$, where

$$\mathcal{L}(u)(x) = \int_y \left(\Phi(u(x-y), x-y, y) - \Phi(u(x), x, y)\right) \nu(dy)$$

for some nonlinear function $\Phi(u, x, y)$ and measure ν. E.g. $\nu(dy) = \frac{dy}{|y|^{a+d}}$. Local analog

$$\mathcal{L}(u)(x) = \sum_{ij} \partial_{ij} \Phi^{ij}(u(x), x).$$

Special cases:

- Spatially homogeneous

$$\mathcal{L}(u) = \int_y \left(\Phi(u(x-y), y) - \Phi(u(x), y)\right) \nu(dy).$$

- Isotropic

$$\mathcal{L}(u) = \int_y \left(\Phi(u(x-y)) - \Phi(u(x))\right) \nu(dy) = \mathcal{L}(\Phi(u)).$$

- Local: Porous medium equation, $m \geq 1$,

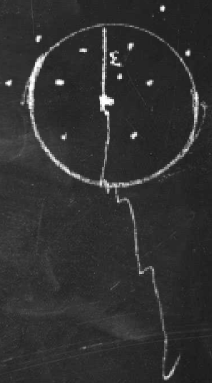
$$\mathcal{L}(u) = \Delta\Phi(u) = \Delta\left(|u|^{m-1} u\right).$$

Aim: Optimal regularity of solutions in (fractional) Sobolev spaces.

Application/Derivation: Interacting (local) diffusions

Consider N particles X_t^i moving randomly according to a Brownian motion. Diffusion depends on the (local) empirical density of particles $\mu^N := \frac{1}{N}\sum_{j=1}^N \delta_{X^j}$. E.g. crowd-avoidance. Get

$$dX_t^i = \sigma(\mu^N, X_t^i)dW_t^i \quad i = 1\dots N.$$



Then μ^N satisfies, with $a = \sigma\sigma^*$,

$$\partial_t \mu^N = \sum_{i,j} \partial_{ij}(a_{ij}(\mu^N, x)\mu^N(x)) + \mathrm{martingale}^N.$$

E.g. $\sigma_{ij}(\mu^N, x) = \delta_{i=j}(K^\varepsilon * \mu^N)^{m_i - 1}(x)$ gives

$$\partial_t \mu^N = \sum_{i} \partial_{ii}((K^\varepsilon * \mu^N)^{m_i-1}(x)\mu^N(x)) + \mathrm{martingale}^N.$$

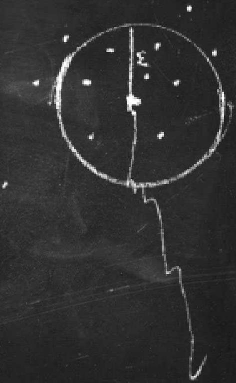
Letting $N \to \infty$ we get $\mu^N \to \mu$, where μ is the (deterministic) solution of the Fokker-Planck equation

$$\partial_t \mu(x) = \sum_{i,j} \partial_{ij}(a_{ij}(\mu, x)\mu(x)).$$

Assume that particle X^i depends only on the number of particles X^j that are sufficiently close, get

$$\partial_t \mu(x) = \sum_{i,j} \partial^2_{ij}(a_{ij}((K^\varepsilon * \mu)(x), x)\mu(x))$$

Localized interaction/moderate interaction: $K^\varepsilon \to \delta$



$$\partial_t \mu(x) = \sum_{i,j} \partial^2_{ij}(a_{ij}(\mu(x), x)\mu(x))$$

Special, anisotropic, homogeneous case: $a_{ij}(\mu) = \mu^{m_i - 1}\delta_{j=i}$,

$$\partial_t \mu(x) = \sum_{i} \partial^2_{ii}(\mu^{m_i}(x))$$

Special, isotropic, homogeneous case: $m_i = m$,

$$\partial_t \mu = \frac{1}{m}\sum_{i} \partial^2_{ii}\mu^m = \frac{1}{m}\Delta\mu^m.$$

Note: $\mu_0 \in L^1(\mathbb{R}^d)$,

Optimal regularity for the (local, homogeneous) porous medium equation

- Scaling and special solutions -

Note, for $m \geq 1$, $u^{[m]} := |u|^m \mathrm{sgn}(u)$,

$$\partial_t u = \frac{1}{m} \Delta u^{[m]}.$$

Consider the Barenblatt solution

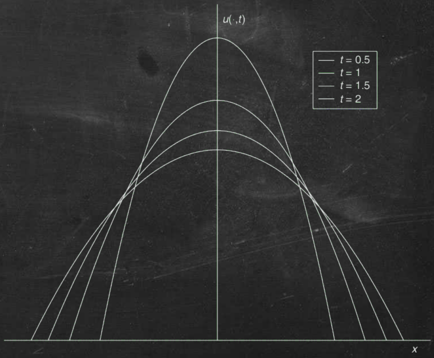
$$u_{BB}(t,x) = t^{-\alpha}(C - k|xt^{-\beta}|^2)_+^{\frac{1}{m-1}}.$$

Then

$$u_{BB} \in L^m([0,T]; \dot{W}^{s,m}(\mathbb{R}^d_x))$$

implies $s < \frac{2}{m}$.

Use $u_{BB}(t,x) = t^{-\alpha} F(xt^{-\beta})$.



Consider

$$\partial_t u = \frac{1}{m}\Delta u^{[m]}.$$

Let $\dot{\mathcal{N}}^{s,p}$ be the homogeneous Nikolskii space ($\dot{\mathcal{N}}^{s,p} = \dot{B}^s_{p,\infty}$).

Theorem (Tadmor, Tao; CPAM 2007, Ebmeyer; JMAA 2005)

Let $u_0 \in L^2(\mathbb{R}^d_x)$, $m \geq 1$. Then

$$\|u\|^{m+1}_{L^{m+1}([0,T];\dot{\mathcal{N}}^{\frac{2}{m+1},m+1}(\mathbb{R}^d_x))} \leq C_m \|u_0\|^2_{L^2_x}.$$

Note: $\frac{2}{m+1} \leq 1$, which is inconsistent with the linear case ($m=1$) and with the optimal regularity of the Barenblatt solution.

By (soft) energy methods may be improved to:

Theorem (G., JEMS, 2021)

Let $\varepsilon > 0$, $m \geq 2$ and $u_0 \in L^{1+\varepsilon}(\mathbb{R}^d_x)$. Then

$$\|u\|^{m+\varepsilon}_{L^{m+\varepsilon}([0,T];\dot{\mathcal{N}}^{\frac{2}{m+\varepsilon},m+\varepsilon}(\mathbb{R}^d_x))} \leq C_{\varepsilon,m} \|u_0\|^{1+\varepsilon}_{L^{1+\varepsilon}_x}.$$

Note: optimal regularity for the Barenblatt solution, but $m \geq 2$ implies $\frac{2}{m+\varepsilon} < 1$.

Problem: How to get to more than one derivative?

Optimal regularity for the porous medium equation

Consider

$$\partial_t u = \frac{1}{m}\Delta u^{[m]} \quad \text{on } (0,T)\times\mathbb{R}^d_x \tag{PME}$$

with $u_0 \in L^1(\mathbb{R}^d_x)$.

Theorem (G., JEMS, 2021)

Let $\varepsilon > 0$, $u_0 \in L^{1+\varepsilon}(\mathbb{R}^d_x)$, $m \geq 1$. Then, for all

$$s \in [0, \frac{2}{m}), \quad p \in [1, m)$$

we have

$$u \in L^p([0,T]; \dot{W}^{s,p}_{loc}(\mathbb{R}^d_x)).$$

In addition, for all $\mathcal{O} \subset\subset \mathbb{R}^d$ there is a constant $C = C(m,p,s,T,\mathcal{O})$ such that

$$\|u\|_{L^p([0,T];\dot{W}^{s,p}(\mathcal{O}))} \leq C\left(\|u_0\|^2_{L^1_x} + 1\right).$$

The kinetic form [Lions, Perthame, Tadmor 1994], [Chen, Perthame; 2003]:For S regular, we have

$$\begin{aligned}\partial_t S(u) &= S'(u)\partial_t u = S'(u)\frac{1}{m}\Delta u^{[m]} \\ &= S'(u)\nabla\cdot(|u|^{m-1}\nabla u) \\ &= \nabla\cdot(S'(u)|u|^{m-1}\nabla u) - \underbrace{S''(u)\nabla u|u|^{m-1}\nabla u}_{\approx S''(u)|\nabla u^{\frac{m+1}{2}}|^2}\end{aligned}$$

Choosing $S_v(u) := (u-v)_+$ we get

$$\partial_t(u-v)_+ = \nabla\cdot(1_{u\geq v}|u|^{m-1}\nabla u) - \delta_{u=v}|\nabla u^{\frac{m+1}{2}}|^2$$

Taking the derivative in v yields

$$\begin{aligned}-\partial_t 1_{v<u(t,x)} &= -\nabla\cdot(\delta_{u(t,x)=v}|u|^{m-1}\nabla u) - \partial_v(\delta_{u(t,x)=v}|\nabla u^{\frac{m+1}{2}}|^2) \\ &= -\nabla\cdot(\delta_{u(t,x)=v}|v|^{m-1}\nabla u) - \partial_v(\delta_{u(t,x)=v}|\nabla u^{\frac{m+1}{2}}|^2) \\ &= -|v|^{m-1}\nabla\cdot(\nabla 1_{v<u(t,x)}) - \partial_v(\underbrace{\delta_{u(t,x)=v}|\nabla u^{\frac{m+1}{2}}|^2}_{=:q \text{ "entropy dissipation measure"}})\end{aligned}$$

Hence, with $\chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}$ we get the kinetic form

$$\partial_t\chi = |v|^{m-1}\Delta_x\chi + \partial_v q.$$

The isotropic case: Recall

$$\partial_t \chi = |v|^{m-1}\Delta_x \chi + \partial_v q \text{ on } (0,T)\times \mathbb{R}^d_x \times \mathbb{R}_v,$$

Variation of constants/Duhamel

$$\chi(t,x,v) = e^{-|v|^{m-1}t\Delta}\chi_0(x,v) + \int_0^t e^{-|v|^{m-1}(t-r)\Delta}\partial_v q(r,x,v)dr.$$

Decompose u in degenerate and non-degenerate part, using that $\chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}$,

$$u(t,x) = \int_v \chi(u(t,x),v) = \underbrace{\int_{|v|\le\delta} \chi(u(t,x),v)}_{u^0(t,x)} + \underbrace{\int_{|v|\ge\delta} \chi(u(t,x),v)}_{u^1(t,x)}.$$

Heat kernel estimates, up to "epsilons"

$$\|u^0\|_{L^\infty_{t,x}} = \left\| \int_{|v|\le\delta} \chi(u(t,x),v) \right\|_{L^\infty_{t,x}} \lesssim \delta,$$

$$\|u^1\|_{L^1_t W^{2,1}_x} \lesssim \delta^{-(m-1)} \| \underbrace{|v|^{-1}q}_{\textit{Singular moments!}} \|_{\mathcal{M}_{t,x,v}}.$$

Real interpolation: Up to "epsilons",

$$\|u\|_{L^m_t W^{\frac{2}{m},m}_x} = \|u\|_{(L^\infty_{t,x}, L^1_t W^{2,1}_x)_{\frac{1}{m},\infty}} < \infty.$$

The anisotropic case: Above, we *essentally* used the isotropy. Anisotropic case needs a different arguments. Indeed, for anisotropic PME, with $m_1 \le ... \le m_d$,

$$\partial_t u = \sum_i \partial^2_{x_i x_i} \frac{1}{m_i} u^{[m_i]} \qquad (\star)$$

kinetic form becomes

$$\partial_t \chi = \sum_i |v|^{m_i - 1} \partial^2_{x_i x_i} \chi + \partial_v q$$

and Fourier transformation of $(\star)$ in time and space

$$\underbrace{i\tau\hat{\chi} - \sum_i |v|^{m_i-1}\xi_i^2 \hat{\chi}}_{=:\mathcal{L}(i\tau,\xi,v)\hat{\chi}} = \partial_v \hat{q}.$$

That is

$$\hat{\chi} = \mathcal{L}^{-1}(i\tau, \xi, v)\partial_v \hat{q}.$$

Idea: Micro-local decomposition of Fourier-space depending on the degeneracy in $|v|$. Localize on Paley-Littlewood blocks $\{\xi : |\xi|^2 \approx 2^j\}$ and micro-locally $\{\xi : |\mathcal{L}(i\tau, \xi, v)| \approx \delta\}$. "uniform truncation property"

Theorem (G., JEMS, 2021)

Let $u_0 \in L^\infty(\mathbb{R}^d_x)$, $m_j \geq 1$, $j = 1, \ldots, d$ *and let* u *be the entropy solution to*

$$\partial_t u = \sum_{i=1}^{d} \partial^2_{x_i x_i} \frac{1}{m_i} u^{[m_i]} \quad \text{on } (0, T) \times \mathbb{R}^d.$$

We set $\underline{m} = \min(\{m_j : j = 1, \ldots, d\})$, $\overline{m} = \max(\{m_j : j = 1, \ldots, d\})$. *Then, for all*

$$s \in \Big[1, \frac{2}{\overline{m}} \left(\frac{\underline{m} - 1}{\overline{m} - 1} \right) \Big), \quad p \in \Big[1, \frac{2\overline{m}}{1 + \overline{m}} \Big),$$

and $\mathcal{O} \subset\subset \mathbb{R}^d$ *there is a constant* $C \geq 0$ *such that*

$$\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \leq C \left(\|u_0\|^{1+}_{L^{1+}_x} + 1 \right).$$

Optimal regularity for nonlocal, inhomogeneous porous medium equation

Recall: (Local) Fokker-Planck equation

$$\partial_t \mu = \sum_{i,j} \partial_{ij}(a_{ij}(\mu(x), x)\mu).$$

How does a nonlocal generalization look like?

Application/Derivation: Consider N particles X_t^i moving randomly according to a Poisson random measure $N(dr, dy)$ on $[0, T] \times \mathbb{R}^d$, with intensity measure $n(dr, dy) := dr\, \nu(dy)$, and ν a Levy measure on $\mathbb{R}^d$. Without interaction would get

$$dX_t^i = \int_y \sigma(X_{r-}^i, y) N^i(dr, dy) \quad i = 1 \ldots N.$$

Diffusion depends on the (local) empirical density of particles $\mu^N := \frac{1}{N} \sum_{j=1, j \neq i}^N \delta_{X^j}$:

$$dX_t^i = \int_y \sigma((\mu_{r-}^N * K^\varepsilon)(X_{r-}^i), X_{r-}^i, y) N^i(dr, dy) \quad i = 1 \ldots N.$$

Propagation of chaos: Letting $N \to \infty$ we get

$$X_t = X_0 + \int_0^t \int_y \sigma((\mu_{r-} * K^\varepsilon)(X_{r-}), X_{r-}, y)\, N(dr, dy),$$

with $\mu = \mathcal{L}(X)$. Localized interaction/moderate interaction: $K^\varepsilon \to \delta$

$$X_t = X_0 + \int_0^t \int_y \sigma(\mu_{r-}(X_{r-}), X_{r-}, y)\, N(dr, dy).$$

Hence, Fokker-Planck equation becomes a nonlocal, anisotropic, degenerate PDE

$$\partial_t \mu = \mathcal{L}(\mu) := \int_y \left(\Phi(\mu(t, x-y), x-y, y) - \Phi(\mu(t,x), x, y)\right) \nu(dy)$$

with $\Phi(\mu, x, y) := \mu \frac{d\sigma(\mu, x, \cdot)_* \nu}{d\nu(\cdot)}(y)$.

Special case (isotropic, $\nu(dy) = \frac{dy}{|y|^{d+a}}$,homogeneous)

$$\partial_t \mu = \Delta^{a/2} \mu^m.$$

For every $m \in (1, \infty)$ and $a \in (0, 2)$ there is suitable bounded and Hölder continuous function $f : [0, \infty) \to \mathbb{R}$, such that

$$u_{bb}(t, x) := t^{-\alpha} f(|x| t^{-\beta}),$$

with $\alpha := \frac{d}{d(m-1)+a}$ and $\beta = \frac{\alpha}{d}$ is a self-similar solution [Vázquez, J. Eur. Math. Soc.; 2014].

Lemma

Let $m \in (1, \infty)$ and $a \in (0, 2)$. Then,

$$u_{BB} \in L^m(0, T; \dot{W}^{\sigma,m}(\mathbb{R}^d))$$

implies $\sigma < \frac{a}{m}$.

Recall

$$\partial_t u = \mathcal{L}(u) := \int_y \left(\Phi(u(t,x-y),x-y,y) - \Phi(u(t,x),x,y)\right)\nu(dy).$$

Informally, the kinetic form reads

$$\begin{aligned}
&\partial_t \chi - \int_{\mathbb{R}^d} (\Phi_v(v,x+y,y)\chi(x+y) - \Phi_v(v,x,y)\chi(x))\nu(dy) \\
&\quad = \partial_v\Big(\chi(x)\int_{\mathbb{R}^d} (\Phi(v,x+y,y) - \Phi(v,x,y))\nu(dy)\Big) + \partial_v q = \partial_v \tilde{q}.
\end{aligned}$$

In Fourier space this becomes

$$\underbrace{(i\tau - \int_{\mathbb{R}^d}(e^{-iy\cdot\xi} - 1)\Phi_v(v,x,y)\nu(dy))}_{:=\mathcal{L}_x(i\tau,\xi,v)}\hat{\chi}(\tau,\xi) = \partial_v \hat{\tilde{q}},$$

where $\mathcal{L}_x(i\tau,\xi,v)$ is a pseudodifferential operator with symbol

$$p_v(x,\xi) := \int_{\mathbb{R}^d}(e^{iy\cdot\xi} - 1)\Phi_v(v,x,y)\nu(dy).$$

Challenges:

- Microlocal analysis of the kinetic operator

$$\mathcal{L}_x(i\tau, \xi, v) = \int_{\mathbb{R}^d} (e^{-iy\cdot\xi} - 1)\Phi_v(v, x, y)\nu(dy).$$

  Extension of the parametrix method "$\mathcal{L}_x^{-1}(i\tau, \xi, v) = ...$" for pseudodifferential operators providing precise bounds in terms of v .
- Since the parametrix is not an exact inverse, lower order terms need to be controlled. Careful interpolation and absorption argument.
- (Possibly non-unique) generalized kinetic solutions.

Special case

$$\partial_t u = \Delta^{a/2} u^m.$$

Corollary (Optimal space-time regularity for nonlocal PME, G., Sauer, 2023)

Let $u_0 \in L^1(\mathbb{R}^d)$, and assume $a \in (0,2)$, $m \in (1,\infty)$. Let $p \in (1,m]$ and

$$\kappa_t := \frac{m-p}{p}\frac{1}{m-1}, \quad \kappa_x := \frac{p-1}{p}\frac{a}{m-1}.$$

Then, for all $\sigma_t \in [0,\kappa_t) \cup 0$ and $\sigma_x \in [0,\kappa_x)$ we have

$$u \in \dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d))$$

and for all $\varepsilon > 0$ there holds

$$\|u\|^p_{\dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d))} \lesssim \|u_0\|^{m+\varepsilon}_{L^1_x} + 1,$$

where the implicit constant depends only on d, a, m, p and ε.

References

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