Finite time extinction for stochastic sign fast diffusion and self-organized criticality.

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Outline

1. Self-organized criticality
2. Finite time extinction for deterministic BTW
3. Finite time extinction for stochastic BTW

Self-organized criticality

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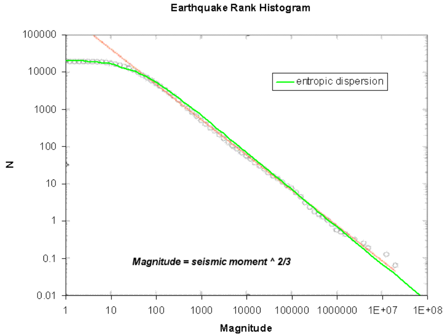
- Many (complex) systems in nature exhibit power law scaling: The number of an event $N(s)$ scales with the event size s as

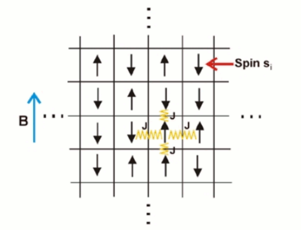
$$N(s) \sim s^{-\alpha}$$

- For example: Earthquakes



Self-organized criticality

- Phase-transitions: The Ising model, ferromagnetism



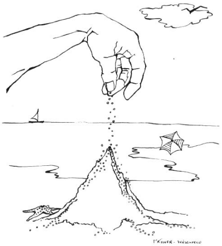
- Critical temperature $T = T_c$:
  - strongly correlated: small perturbations can have global effects
  - no specific length scale (complex system, criticality)
- Observe: For $T = T_c$, power-law scaling for $N(s)$ being the number of $+1$ clusters of size s.

Self-organized criticality

- Ising model needs precise tuning $T = T_c$ to display power law scaling
- How can this occur in nature?
- Idea of self-organized criticality: [Bantay, Ianosi; Physica A, 1992]

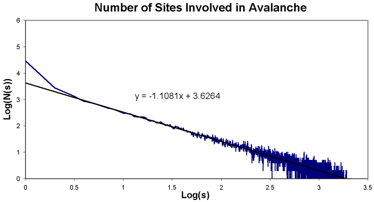
"Criticality" refers to the power-law behavior of the spatial and temporal distributions, characteristic of critical phenomena.
"Self-organized" refers to the fact that these systems naturally evolve into a critical state without any tuning of the external parameters, i.e. the critical state is an attractor of the dynamics.

- Bak, Tang, Wiesenfeld: Sandpile as a toy model of self-organized criticality



Sandpiles

- Two scales: Slow energy injection (adding sand), fast energy diffusion (avalanches)
- Criticality: No typical avalanche size, local perturbation may have global effects
- Power law scaling: $N(s)$ is the number of valances of size s.



The stochastic BTW model

- In [Díaz-Guilera; EPL (Europhysics Letters), 1994], [Giacometti, Diaz-Guilera; Phys. Rev. E, 1998], [Díaz-Guilera; Phys. Rev. A, 1992] the following model for self-organized criticality (SOC) was introduced:

$$dX_t \in \Delta H(X_t - X^c) + B(X_t - X^c)dW_t,$$

with appropriate diffusion coefficients B and Dirichlet boundary conditions.

- We study linear multiplicative noise, i.e.

$$dX_t = \in \Delta H(X_t - X^c) + \sum_{k=1}^{N} f_k(X_t - X^c)d\beta_t^k.$$

with zero Dirichlet boundary conditions.

- Question: Does the diffusion end in finite time?

Finite time extinction and SOC

- We will restrict to the supercritical case, i.e. supposing $x_0 \geq X^c$.
- Substituting $X \to X - X^c$ and using $X \geq 0$ yields

$$dX_t \in \Delta \mathrm{sgn}(X_t) + \sum_{k=1}^{N} f_k X_t d\beta_t^k,$$
$$X(0) = x_0$$

with $x_0 \geq 0$ and zero Dirichlet boundary conditions:

$$\mathrm{sgn}(X(t,\xi)) \ni 0, \quad \text{on } \partial\mathscr{O}.$$

- Informally:
$$\Delta \mathrm{sgn}(X) = 2\delta_0(X)\Delta X + \mathrm{sgn}''(X)|\nabla X|^2.$$
- Diffusion ends in finite time = Finite time extinction.

The stochastic BTW model

- Recall:

$$dX_t \in \Delta \text{sgn}(X_t) + \sum_{k=1}^{N} f_k X_t d\beta_t^k, \qquad \text{(BTW)}$$

  with zero Dirichlet boundary conditions.
- Finite time extinction can be reformulated in terms of the extinction time

$$\tau_0(\omega) := \inf\{t \geq 0 | X_t(\omega) = 0, \text{ a.e. in } \mathcal{O}\}.$$

  We distinguish the following concepts:

  (F1) Extinction with positive probability for small initial conditions: $\mathbb{P}[\tau_0 < \infty] > 0$, for small $X_0 = x_0$.

  (F2) Finite time extinction: $\mathbb{P}[\tau_0 < \infty] = 1$, for all $X_0 = x_0$.

- (F2) for (BTW) has been addressed but left open in: [V. Barbu, MMAS, 2013], [M. Röckner, F-Y. Wang, JLMS, 2013], [V. Barbu, G. Da Prato, M. Röckner, JMAA, 2012], [V. Barbu, M. Röckner, CMP, 2012], [V. Barbu, G. Da Prato, M. Röckner, CMP, 2009], [V. Barbu, G. Da Prato, M. Röckner, CRMAS, 2009]

Main result

Theorem (Main result)

Let $x_0 \in L^\infty(\mathcal{O})$, X be the unique variational solution to BTW and let

$$\tau_0(\omega) := \inf\{t \geq 0 | X_t(\omega) = 0, \text{ for a.e. } \xi \in \mathcal{O}\}.$$

Then finite time extinction holds, i.e.

$$\mathbb{P}[\tau_0 < \infty] = 1.$$

For every $p > \frac{d}{2} \vee 1$, the extinction time $\tau_0(\omega)$ may be chosen uniformly for x_0 bounded in $L^p(\mathcal{O})$.

Finite time extinction for deterministic PDE

Finite time extinction for deterministic PDE

Finite time extinction for singular ODE

- Consider the singular ODE

$$\dot{f} = -cf^{\alpha}, \quad \alpha \in (0,1), \ c > 0.$$

- Then:

$$(f^{1-\alpha})' = -(1-\alpha).$$

- We obtain

$$f^{1-\alpha}(t) = f^{1-\alpha}(0) - (1-\alpha)ct$$

which implies finite time extinction.

Finite time extinction and SOC

- [Diaz, Diaz; CPDE, 1979] finite time extinction (FTE) was first proven for

$$\frac{\partial}{\partial t}X(t,\xi) \in \Delta \mathrm{sgn}(X(t,\xi)).$$

- In [Barbu; MMAS, 2012] another (more robust) approach based on energy methods was introduced.

Finite time extinction and SOC

- Informally the proof boils down to a combination of an L^1 and an L^∞ estimate of the solution:
- Informal L^∞ estimate:

$$\|X(t)\|_\infty \leq \|x_0\|_\infty, \quad \forall t \geq 0.$$

- Informal L^1-estimate:

$$\begin{aligned}
\partial_t \int_{\mathscr{O}} |X(t,\xi)| d\xi &= \int_{\mathscr{O}} \mathrm{sgn}(X(t,\xi)) \Delta \mathrm{sgn}(X(t,\xi)) d\xi \\
&= -\int_{\mathscr{O}} |\nabla \mathrm{sgn}(X(t,\xi))|^2 d\xi \\
&\leq -\left(\int_{\mathscr{O}} |\mathrm{sgn}(X(t,\xi))|^p d\xi \right)^{\frac{2}{p}} \\
&\leq -(|\{\xi | X(t,\xi) \neq 0\}|)^{\frac{2}{p}},
\end{aligned}$$

for some (dimension dependent) $p > 2$. Note: $\frac{2}{p} < 1$.

Finite time extinction and SOC

- Observe

$$\int_{\mathscr{O}} |X(t,\xi)|d\xi \leq \|X(t)\|_\infty |\{\xi | X(t,\xi) \neq 0\}|.$$
$$\leq \|x_0\|_\infty |\{\xi | X(t,\xi) \neq 0\}|.$$

- Using this above gives

$$\partial_t \int_{\mathscr{O}} |X(t,\xi)|d\xi \leq -\frac{1}{\|x_0\|_\infty^{\frac{2}{p}}} \left(\int_{\mathscr{O}} |X(t,\xi)|d\xi \right)^{\frac{2}{p}}.$$

- We are left with the singular ODE

$$\dot{f} = -cf^\alpha, \quad \alpha \in (0,1),\ c > 0$$

for which we have seen that finite time extinction holds.

Finite time extinction for stochastic BTW

Finite time extinction for stochastic BTW

Transformation

- Recall:

$$dX_t \in \Delta \mathrm{sgn}(X_t) + \sum_{k=1}^{N} f_k X_t \, d\beta_t^k,$$

- Our approach to FTE will be based on considering the following transformation: Set $\mu_t := \sum_{k=1}^N f_k \beta_t^k$, $\tilde{\mu} := \sum_{k=1}^N f_k^2$ and $Y_t := e^{-\mu_t} X_t$. An informal calculation shows

$$\partial Y_t \in e^{\mu_t} \Delta \mathrm{sgn}(Y_t) - \tilde{\mu} Y_t. \qquad (*)$$

- Compare the deterministic setting:

$$\partial Y_t \in \Delta \mathrm{sgn}(Y_t).$$

Outline of the proof

- There are two main ingredients of the proof:
  1. A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
  2. An energy inequality for a weighted L^1-norm.
- On an intuitive level the arguments become clear by approximating

$$r^{[m]} := |r|^{m-1} r \to \text{sgn}, \quad \text{for } m \downarrow 0.$$

  To make these arguments rigorous, in fact a different (non-singular, non-degenerate) approximation of sgn is used.
- In the following let Y_t be a solution to

$$\partial_t Y_t \in e^{\mu_t} \Delta Y_t^{[m]} - \tilde{\mu} Y_t.$$

Step 1: Informal L^p bound

- **Step 1:** A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
- We may informally compute for all $p \geq 1$:

$$\begin{aligned}\partial_t \int_{\mathcal{O}} |Y_t|^p d\xi =& p \int_{\mathcal{O}} Y_t^{[p-1]} e^{\mu_t} \Delta Y_t^{[m]} d\xi \\ =& -\frac{4(p-1)mp}{(p+m-1)^2} \int_{\mathcal{O}} e^{\mu_t} \left(\nabla |Y_t|^{\frac{p+m-1}{2}}\right)^2 d\xi \\ &+ \frac{pm}{p+m-1} \int_{\mathcal{O}} |Y_t|^{p+m-1} \Delta e^{\mu_t} d\xi .\end{aligned}$$

- Taking $p > 1$ and then $m \to 0$ we may "deduce" from this

$$\partial_t \int_{\mathcal{O}} |Y_t|^p d\xi \leq 0.$$

Step 2: Informal "L^1" bound

- **Step 2:** An energy inequality for a weighted L^1-norm.

$$\partial_t \int_{\mathcal{O}} |Y_t|^p d\xi = -\frac{4(p-1)mp}{(p+m-1)^2} \int_{\mathcal{O}} e^{\mu_t} \left(\nabla |Y_t|^{\frac{p+m-1}{2}} \right)^2 d\xi$$
$$+ \frac{pm}{p+m-1} \int_{\mathcal{O}} |Y_t|^{p+m-1} \Delta e^{\mu_t} d\xi, \ p \geq 1.$$

- Choose $p = m+1$ and let $m \to 0$. We obtain

$$\partial_t \int_{\mathcal{O}} |Y_t| d\xi = -\int_{\mathcal{O}} e^{\mu_t} \left(\nabla \mathrm{sgn}(Y_t) \right)^2 d\xi + \frac{1}{2} \int_{\mathcal{O}} \Delta e^{\mu_t} d\xi$$

- Recall: deterministic case

$$\partial_t \int_{\mathcal{O}} |Y_t| d\xi = -\int_{\mathcal{O}} |\nabla \mathrm{sgn}(Y_t)|^2 d\xi.$$

Step 2: Informal "L^1" bound

Key trick: Use a weighted L^1-norm

- Let φ be the classical solution to

$$\Delta\varphi = -1, \quad \text{on } \mathcal{O}$$
$$\varphi = 1, \quad \text{on } \partial\mathcal{O}.$$

  Note $1 \leq \varphi \leq \|\varphi\|_\infty =: C_\varphi$.

- We informally compute

$$\partial_t \int_{\mathcal{O}} \varphi |Y_t| d\xi = -\int_{\mathcal{O}} \varphi e^{\mu_t} (\nabla \mathrm{sgn}(Y_t))^2 \, d\xi + \frac{1}{2} \int_{\mathcal{O}} \Delta(\varphi e^{\mu_t}) d\xi.$$

- Note

$$\Delta(\varphi e^{\mu_t}) = -e^{\mu_t} + 2\nabla\varphi \cdot \nabla e^{\mu_t} + \varphi \Delta e^{\mu_t}$$

  has a negative sign for small times ($e^{\mu_t} \approx 1$)!

- Shift the initial time

$$\partial_t \int_{\mathcal{O}} e^{-\mu_s} \varphi |Y_t| d\xi = -\int_{\mathcal{O}} e^{\mu_t - \mu_s} \varphi (\nabla \mathrm{sgn}(Y_t))^2 \, d\xi + \frac{1}{2} \int_{\mathcal{O}} \mathrm{sgn}(Y_t)^2 \Delta e^{\mu_t - \mu_s} \varphi d\xi$$

Thanks

Thanks!