Fluctuations in continuum

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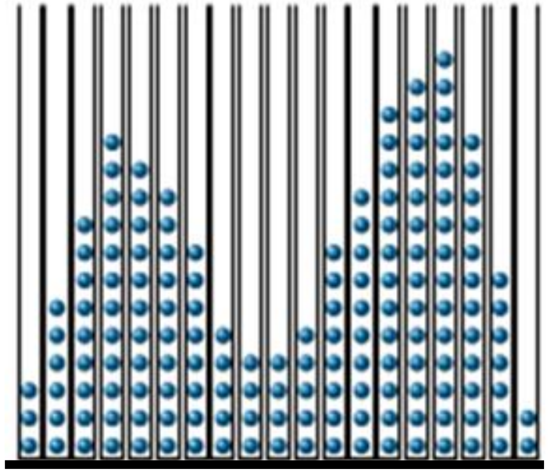
Agenda

- Microscopic – Macroscopic – Mesoscopic: Stochastic partial differential equations
- How to model the noise?
- From large deviations to gradient flows
- Order of approximation
- Mathematical challenges

Microscopic – Macroscopic – Mesoscopic

Microscopic scale: Particles



Gridsize = $\frac{1}{N}$

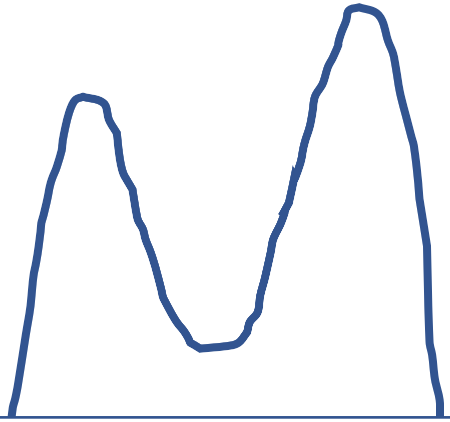
$$\frac{d}{dt}X^i(t) = b(X^i(t), X(t)) + \sigma(X^i(t), X(t))\dot{N}^i(t), \quad i = 1 \dots N$$

$\eta(t,x)$ = number of particles at time t in box x

$N \to \infty$

$\mu^N(t,x)$ = empirical density field

Macroscopic scale: PDEs



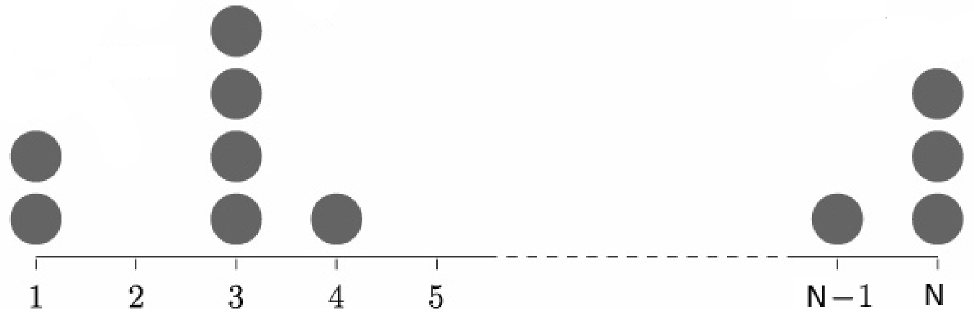
Think $\Delta\bar{\rho}(t,x) = \partial_{xx}\bar{\rho}(t,x)$

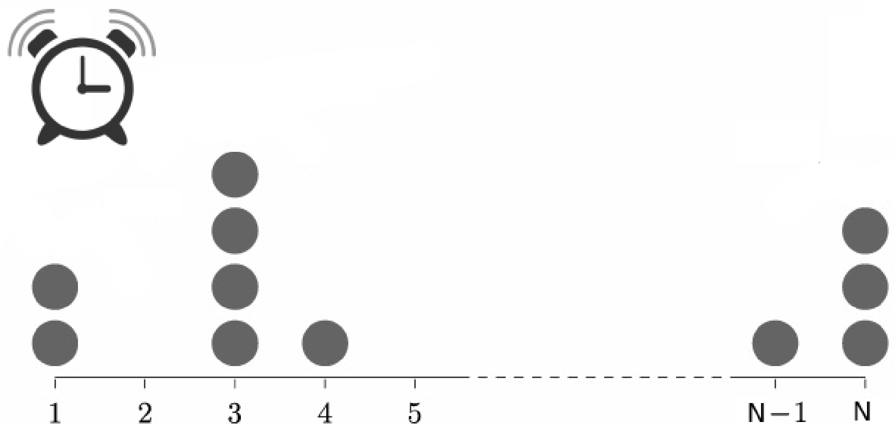
$$\partial_t \bar{\rho}(t,x) = \Delta\bar{\rho}(t,x)$$

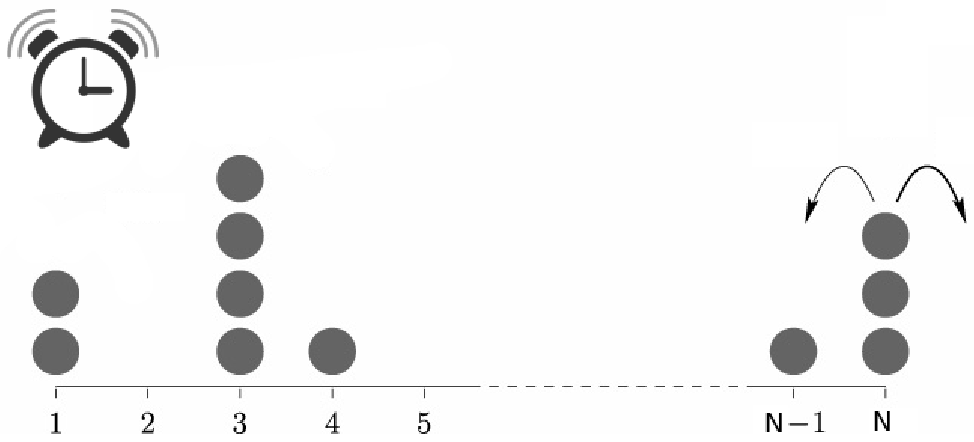
Zero range process – An example of a stochastic particle system

What is?

- N boxes
- Finite number of particles in each box
- When clock rings, a random particle jumps
- Jump rate may depend on number of particles in the box

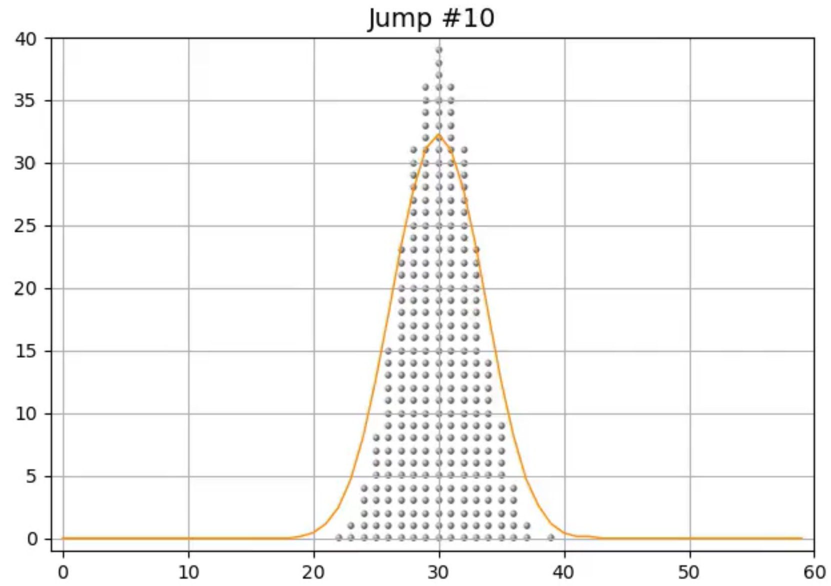


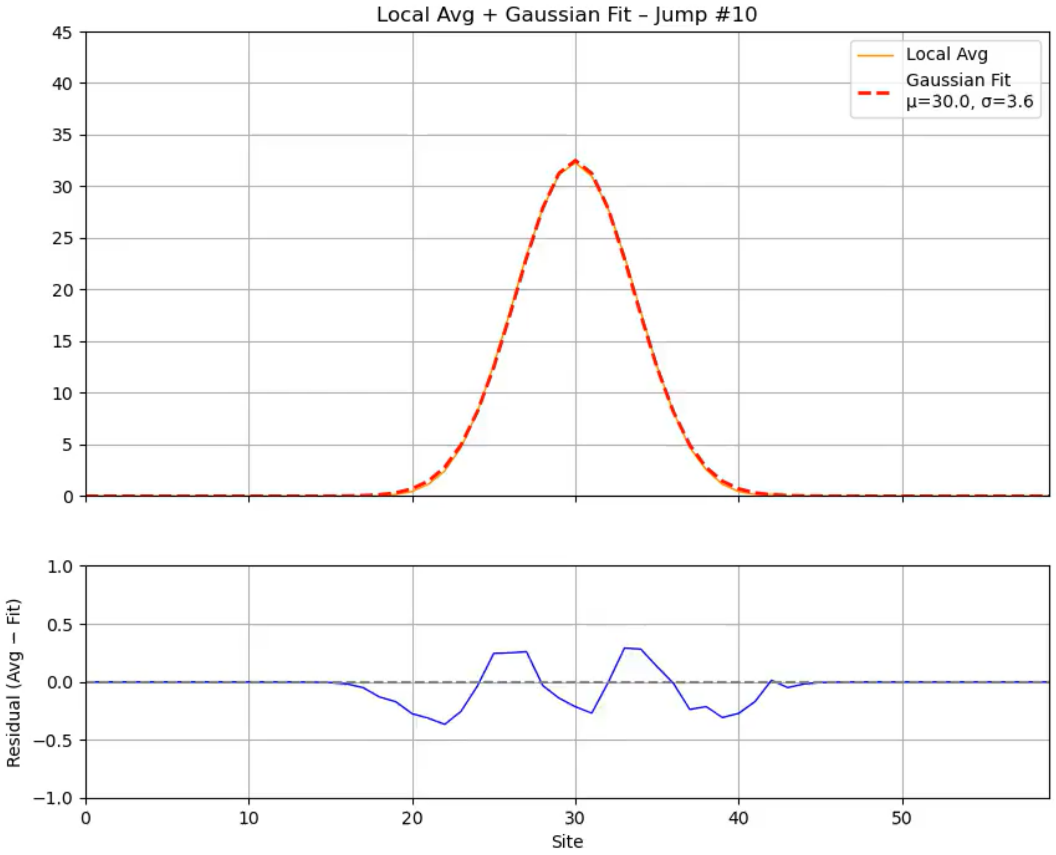




Zero range process – An example of a stochastic particle system

Toy model: Independent particles



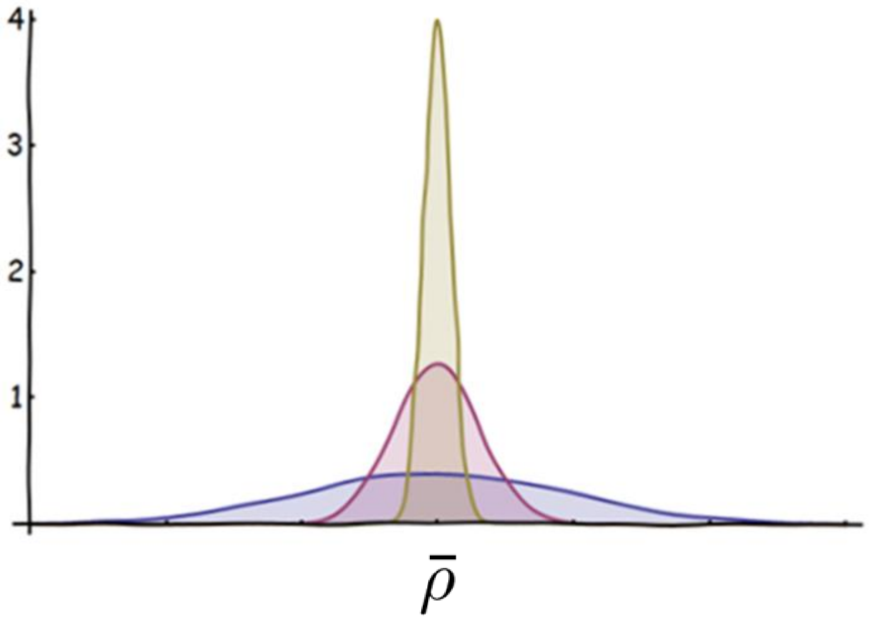


In the limit $N \to \infty$ we get

$$\mu^N(t,x) \rightharpoonup^* \bar{\rho}(t,x)$$

with

$$\partial_t \bar{\rho}(t,x) = \Delta \bar{\rho}(t,x)$$



Limitations: No information about fluctuations is preserved

- Limited order of convergence

$$d(\mu^N, \bar{\rho}) \approx N^{-d/2}$$

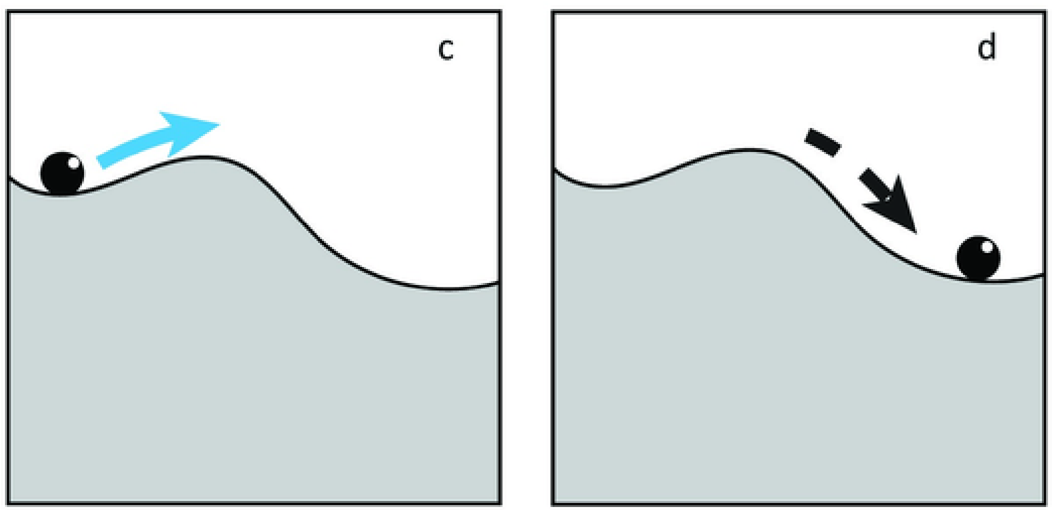
- Large deviations

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\,\mathcal{I}(\rho)} \quad N \text{ large}$$

Zero range process – An example of a stochastic particle system

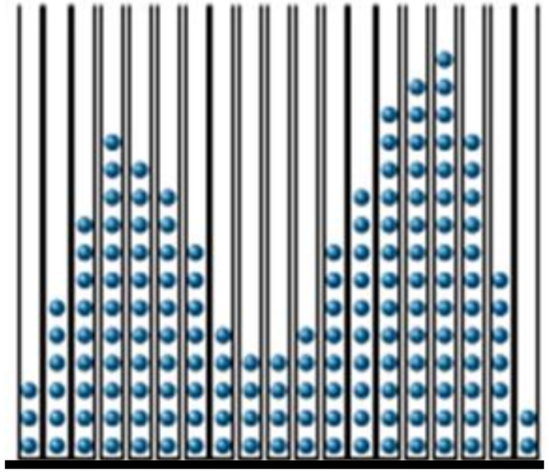
Relevance of fluctuations:

- Rare but catastrophic events:
  - Failure of mechanical devices
  - Earthquakes
- Tipping points



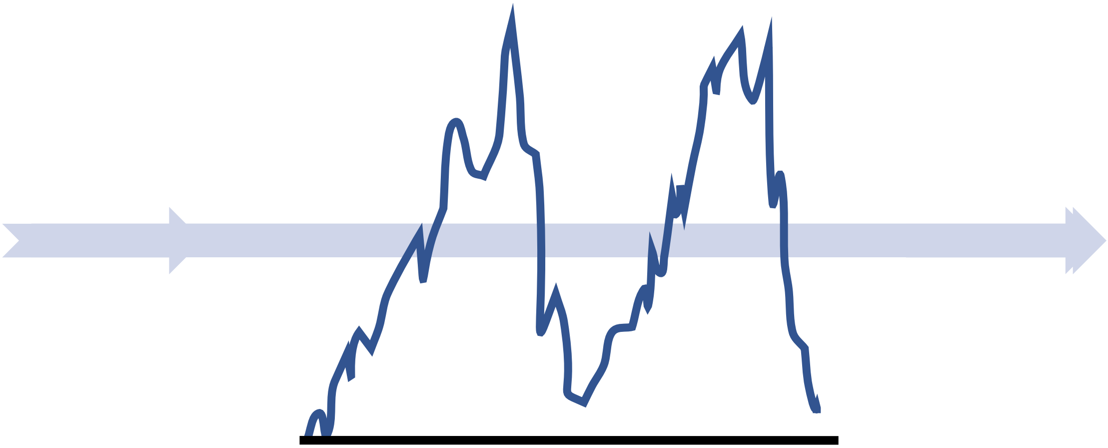
Microscopic – Macroscopic – Mesoscopic

Microscopic scale: Particles



Gridsize = $\frac{1}{N}$

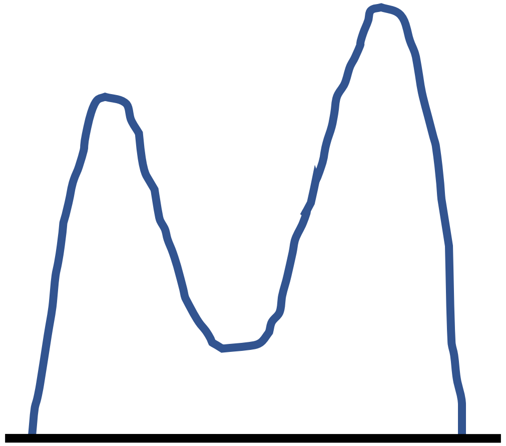
Mesoscopic scale: Conservative SPDEs



Fluctuation correction

$$\partial_t \rho(t,x) = \Delta \rho(t,x) + \underbrace{\text{"noise"}}_{?}$$

Macroscopic scale: PDEs



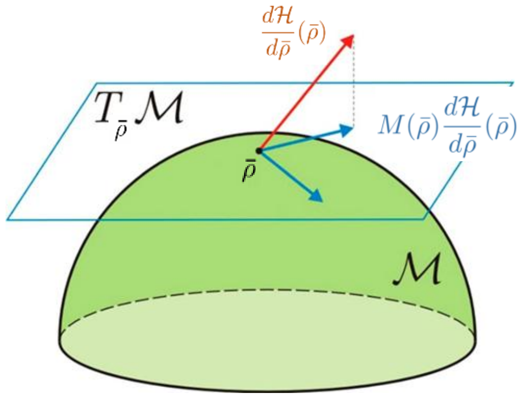
Mean dynamics

$$\partial_t \bar{\rho}(t,x) = \Delta \bar{\rho}(t,x)$$

How to model the noise?

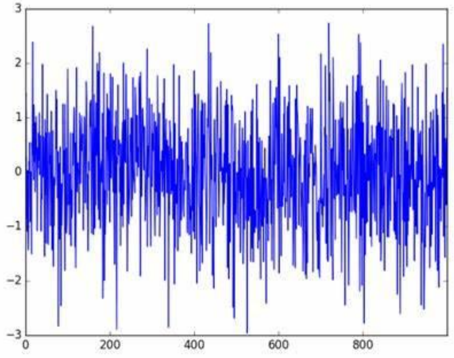
How to choose the noise? Fluctuations, gradient flow structures and large deviations.

$$\partial_t \rho(t,x) = \Delta \rho(t,x) + \underbrace{\text{"noise"}}_{?}$$



Reinterpret PDE as a gradient flow of entropy $\mathcal{H}$ on a manifold $\mathcal{M}$

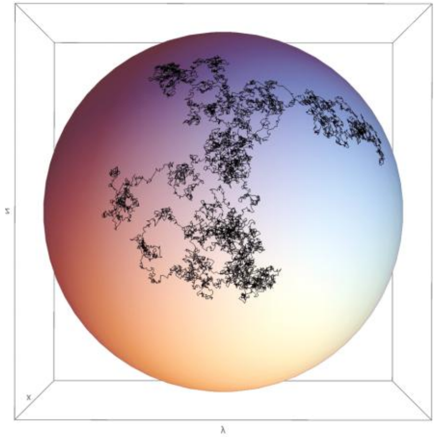
$$\partial_t \bar{\rho} = \Delta \bar{\rho} =_? -\nabla_{\mathcal{M}} \mathcal{H}(\bar{\rho}) = -M(\bar{\rho}) \frac{d\mathcal{H}}{d\bar{\rho}}(\bar{\rho})$$



Fluctuation-dissipation principle: Stochastic gradient flow

$$\partial_t \rho = -M(\rho)\frac{d\mathcal{H}}{d\rho}(\rho) + M^{\frac{1}{2}}(\rho) \diamond \xi,$$

where ξ is space-time white noise.



Conclusion: Different gradient flow structures lead to different SPDEs.

Are there gradient flow interpretations?

Gradient flows for the heat equation?

Brezis ['71] (even for nonlinear diffusions)

$$\partial_t \rho = \Delta \rho = \Delta \frac{D\mathcal{H}(\rho)}{D\rho}.$$

$$\mathcal{M} = H^{-1} = (H^1)^*,\ M(\rho) = -\Delta,\ \mathcal{H}(\rho) = \frac{1}{2}\int \rho^2$$

Additive noise:

$$\partial_t \rho = -M(\rho)\frac{d\mathcal{H}}{d\rho}(\rho) + M^{\frac{1}{2}}(\rho) \diamond \xi = \Delta \rho + \nabla \diamond \xi.$$

Otto ['01]:

$$\partial_t \rho = \Delta \rho = \nabla \cdot (\rho \underbrace{\nabla \log(\rho)}_{=\frac{1}{\rho}\nabla \rho}).$$

$$\mathcal{M} = \mathcal{P}(\mathbb{T}^d),\ M(\rho) = -\nabla \cdot (\rho \nabla \cdot),\ \mathcal{H}(\rho) = \int \log(\rho)\rho \text{ Boltzmann entropy},$$

Dean-Kawasaki equation:

$$\partial_t \rho = -M(\rho)\frac{d\mathcal{H}}{d\rho}(\rho) + M^{\frac{1}{2}}(\rho) \diamond \xi = \Delta \rho + \nabla(\sqrt{\rho} \diamond \xi).$$

Which gradient flow should we choose? Depends on the physical context.

The Entropy Dissipation Inequality (EDI) formulation of gradient flows

Consider

$$\partial_t \rho = -\nabla_{\mathcal{M}} \mathcal{H}(\rho). \qquad (\star)$$

Then

$$\partial_t \mathcal{H}(\rho) = -(\partial_t \rho, \frac{D\mathcal{H}}{D\rho})_{M(\rho)} \geq -|\partial_t \rho|_{M(\rho)} |\frac{D\mathcal{H}}{D\rho}|_{M(\rho)} \geq -\frac{1}{2}|\frac{D\mathcal{H}}{D\rho}|^2_{M(\rho)} - \frac{1}{2}|\partial_t \rho|^2_{M(\rho)}$$

Integrating time from 0 to T

$$\mathcal{I}(\rho) = \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\int_0^T |\frac{D\mathcal{H}}{D\rho}|^2_{M(\rho)} + \frac{1}{2}\int_0^T |\partial_t \rho|^2_{M(\rho)} \geq 0$$

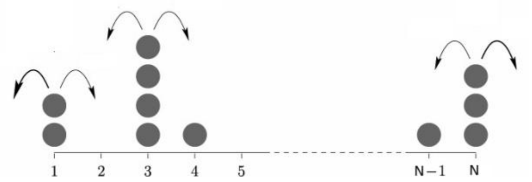
with equality iff ρ solves (⋆).

Consequence: ρ is a gradient flow for (⋆) iff

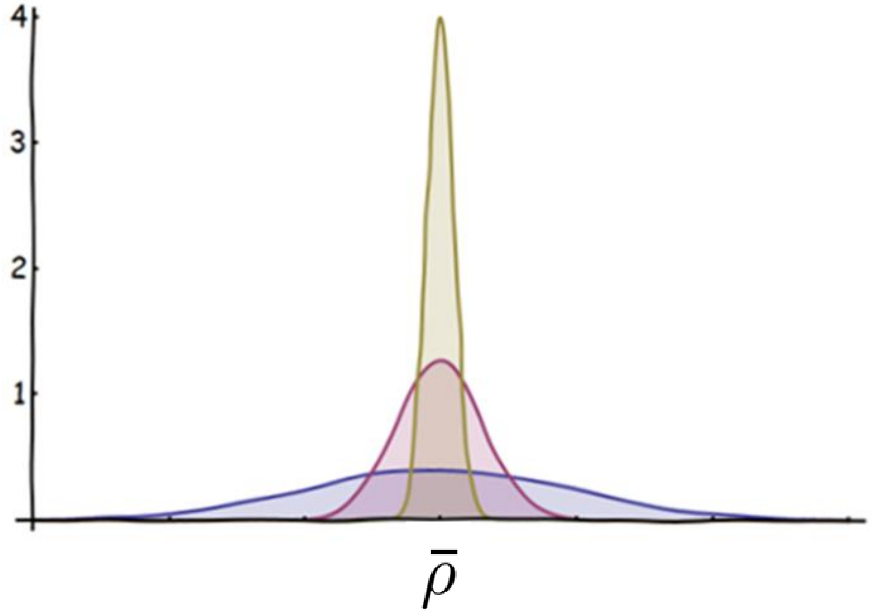
$$\rho = \mathrm{argmin}_\rho \mathcal{I}(\rho).$$

How to relate this to particle systems?

We have the law of large numbers



$$\mu^N(t,x) \rightharpoonup^* \bar{\rho}(t,x)$$
$$\partial_t \bar{\rho}(t,x) = \Delta \bar{\rho}(t,x)$$



Deviations from this mean behavior?

Large deviations: Given any function ρ estimate

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\mathcal{I}(\rho)} \quad N \text{ large}$$

for some rate function $\mathcal{I}$.

This characterizes the solution $\bar{\rho}$ as

$$\bar{\rho} = \operatorname{argmin}_\rho \mathcal{I}(\rho).$$

This fixes the $\mathcal{I}$ in the EDI formulation! If we can rewrite $\mathcal{I}$ in the form

$$\mathcal{I}(\rho) = \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\int_0^T |\frac{D\mathcal{H}}{D\rho}|^2_{M(\rho)} + \frac{1}{2}\int_0^T |\partial_t \rho|^2_{M(\rho)} \geq 0$$

Conclusion in toy model example: $\partial_t \rho = -M(\rho)\dfrac{d\mathcal{H}}{d\rho}(\rho) + M^{\frac{1}{2}}(\rho) \diamond \xi = \Delta\rho + \nabla(\sqrt{\rho} \diamond \xi).$

Order of approximation

Error in approximation by (deterministic) PDE

$$\partial_t \bar{\rho}(t,x) = \Delta \bar{\rho}(t,x)$$

is

$$d(\mu^N, \bar{\rho}) \approx N^{-d/2}$$

Error in approximation by stochastic PDE

$$\partial_t \rho = \Delta \rho + \frac{1}{N^{d/2}} \nabla \cdot (\sqrt{\rho} \diamond \xi).$$

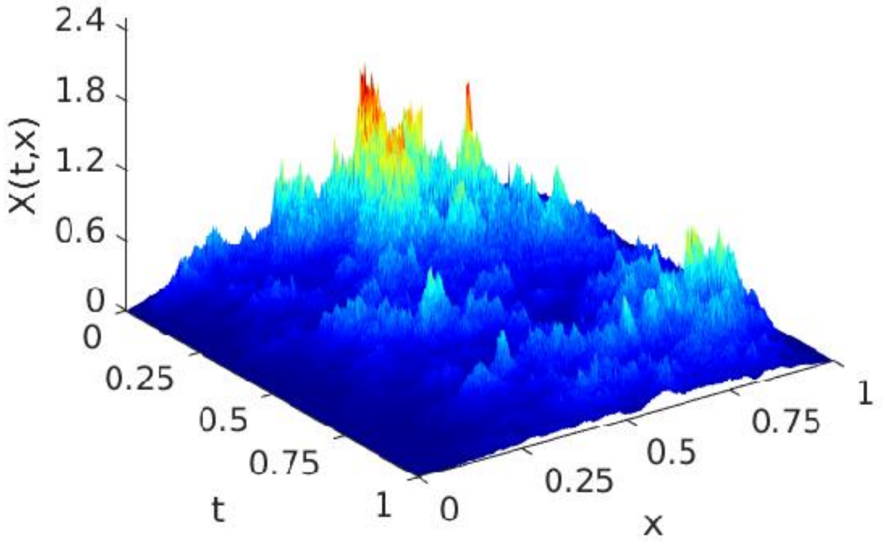
Is at most

$$d(\mu^N, \rho) \approx N^{-d}$$

Mathematical challenges

Conservative SPDEs: Mathematical challenges

$$\partial_t \rho = \Delta\rho + \tfrac{1}{2}\nabla \cdot (\sigma(\rho) \diamond \xi)$$



One-dimensional space-time white noise ξ has Hölder regularity $-\frac{3}{2}-$

Hence, the term $\nabla \cdot (\sigma(\rho) \diamond \xi)$ has Hölder regularity $-\frac{5}{2}-$

With the heat operator solution ρ has Hölder regularity $-\frac{1}{2}-$

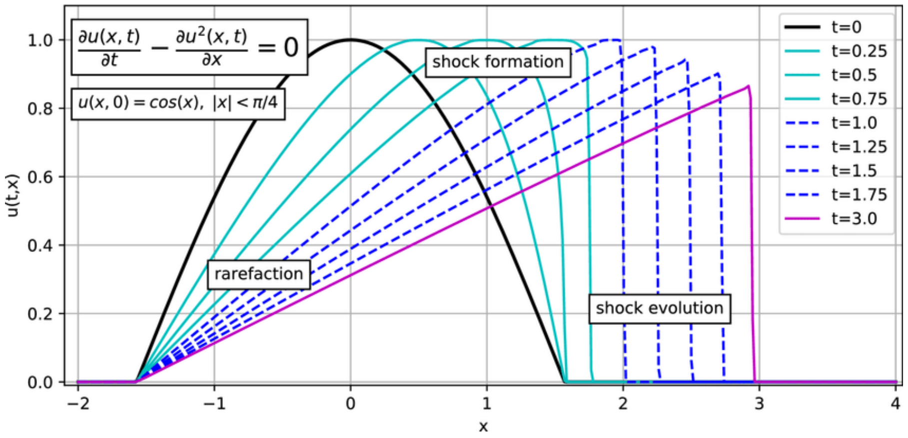
Meaning of $\sigma(\rho)$? Irregularity, „ultra-violet catastrophe“,
renormalization necessary = subtracting blow ups

Conservative SPDEs: Mathematical challenges

$$\partial_t \rho = \Delta \rho + \tfrac{1}{2}\nabla \cdot (\sigma(\rho) \diamond \xi)$$

Nonlinear conservation law: Occurrence of shocks

$$\partial_t \rho = \tfrac{1}{2}\partial_x \rho^2$$



Viscous conservation laws: No shocks

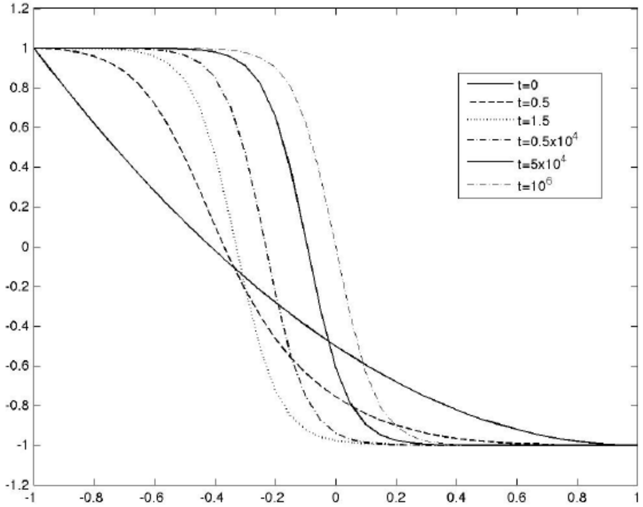
$$\partial_t \rho = \partial_{xx}\rho + \tfrac{1}{2}\partial_x \rho^2$$

On small scales

$$\tilde{\rho}(t,x) = \rho(\tau^2 t, \tau x)$$

$$\partial_t \tilde{\rho}(t,x) = \tau \partial_t \rho(\tau^2 t, \tau x) = \partial_{xx}\rho + \tfrac{1}{2}\tau^{\frac{1}{2}}\partial_x \rho^2$$

diffusion dominates.



Stochastic viscous conservation laws: Do shocks still appear?

$$\partial_t \rho = \partial_{xx}\rho + \tfrac{1}{2}\partial_x \rho^2 \circ \xi$$

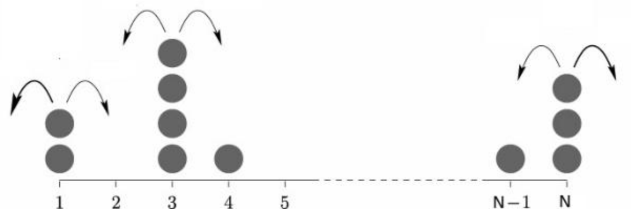
Rescaling invariant!

$$\partial_t \tilde{\rho}(t,x) = \tau \partial_t \rho(\tau t, \tau x) = \partial_{xx}\tilde{\rho} + \tfrac{1}{2}\partial_x \tilde{\rho}^2 \circ \xi$$

From large deviations to PDE with irregular coefficients

Large deviations principle

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\mathcal{I}(\rho)} \quad N \text{ large}$$



has rate function (think $\alpha = 1$)

$$\mathcal{I}(\rho) = \inf\left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dxds : \ g \in L^2_{t,x}, \ \underbrace{\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)}_{\text{"skeleton equation"}} \right\}$$

To prove large deviations principle, need to show the existence and uniqueness of solutions to the skeleton equation

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g), \quad \text{with } g \in L^2_{t,x}.$$

Very irregular coefficients!

From large deviations to PDE with irregular coefficients

Regularity for solutions to the PDE

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g), \quad \text{with } g \in L^2_{t,x}.$$

Zoom into local scales: Which operator dominates?

Energy criticality: Even on small scales both diffusion and convection are present.

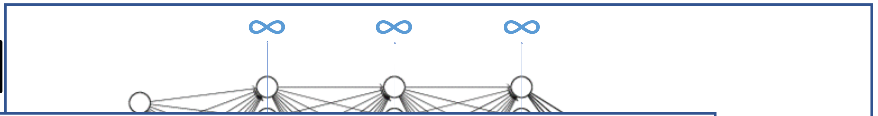
Time-reversal: Let ρ_r be the time reversed solution, then

$$\begin{aligned} \partial_t \rho_r &= -\Delta \rho_r^\alpha + \nabla \cdot (\rho_r^{\alpha/2} \mathcal{T} g) \\ &= \Delta \rho_r^\alpha - \nabla \cdot \rho_r^{\frac{\alpha}{2}} (g_r). \end{aligned}$$

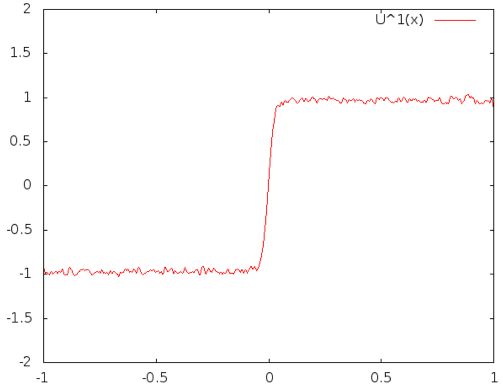
Hence, the only available energy estimate is the entropy dissipation estimate. No other L^p_x estimates.

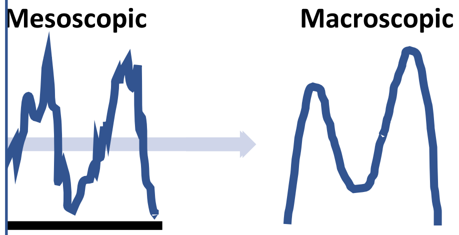
Indeed, the diffusion does not dominate!

Fluctuations in compl

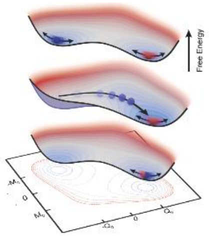


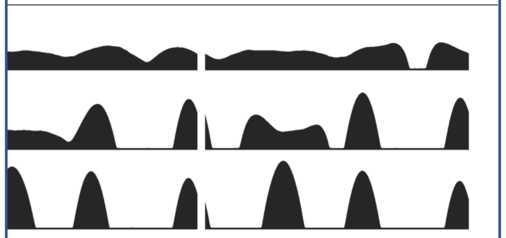
Fluctuating interfaces





Metastability in mean field systems





Fluctuating mean field systems

Stochastic fluid dynamics

$$\partial_t \rho = \nabla \cdot F(\rho, D\rho, D^3\rho) + \nabla \cdot (\sigma(\rho) \diamond \xi).$$

References

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