Part 1: Introduction to (stochastic) scalar conservation laws

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Outline

Contents of the course:

1. Deterministic theory (entropy solutions, kinetic solutions, quasi-solutions)
2. Stochastic well-posedness theory
3. Well-posedness by noise for scalar conservation laws (spatially irregular fluxes)
4. Regularization by noise for scalar conservation laws

Part I

Part I

- Continuum mechanics: Consider some test volume $V \subseteq \mathbb{R}^d$. The change of some physical quantity with density ρ is

$$\frac{d}{dt}\int_V \rho(t,x)\,dx = -\int_{\partial V} q(t,x)\cdot n_V(x)\,d\mathscr{H}^{d-1}(x) + \int_V r(t,x)\,dx,$$

where q is the flux across the boundary V and r the rate of change of ρ in V.

- Gauss's theorem

$$\frac{d}{dt}\int_V \rho(t,x)\,dx = \int_V -\operatorname{div} q(t,x) + r(t,x)\,dx.$$

- Since V is arbitrary we get

$$\frac{d}{dt}\rho(t,x) + \operatorname{div} q(t,x) = r(t,x) \quad \text{on } (0,T)\times\mathbb{R}^d.$$

- Assume that q is given as a function of ρ: $q(t,x) = A(\rho(t,x))$. Then

$$\frac{d}{dt}\rho(t,x) + \operatorname{div} A(\rho(t,x)) = r(t,x).$$

Example: The equation of road traffic

- Let ρ be the density of cars, v the (mean) velocity
- Conservation of 'mass'

$$\partial_t \rho + \partial_x q = 0 \quad \text{on } (0,T) \times \mathbb{R},$$

  where $q = \rho v$ is the flux.

- Velocity according to the traffic conditions:
  - $v = V(\rho)$, where V is the speed limit if ρ is small.
  - $\rho \mapsto V(\rho)$ is decreasing, $V(\rho_m) = 0$ at saturation value ρ_m.
- We obtain

$$\partial_t \rho + \partial_x (\rho V(\rho)) = 0 \quad \text{on } (0,T) \times \mathbb{R}.$$

- Consider
$$\partial_t u + \partial_x A(u) = 0, \quad u(0) = u_0. \qquad (\star)$$
- Method of characteristics:
$$X_t^x = A'(u_0(x))t + x$$
then
$$u(t, X_t^x) = u_0(x).$$
- Implicit formula: Given $(t, y) \in [0, T] \times \mathbb{R}$ we find x such that
$$y = A'(u_0(x))t + x.$$
Then put $u(t, y) = u_0(x)$.

Proposition (Dafermos; 2010)

Let $u_0 \in Lip(\mathbb{R})$. Set
$$T^* = \begin{cases} \infty & \text{if } A'(u_0) \text{ non-decreasing} \\ -\left(\inf_{x\in\mathbb{R}} \frac{d}{dx} A'(u_0(x))\right)^{-1} & \text{otherwise.} \end{cases}$$
Then $(\star)$ has a unique classic solution in $(0, T^*) \times \mathbb{R}$ that cannot be extended beyond T^*.

Consider the Burgers equation

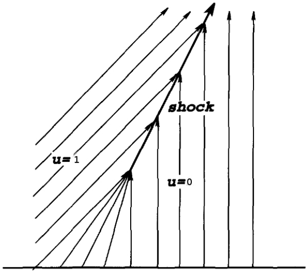
$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0 \quad \text{on } (0,T)\times\mathbb{R}$$
$$u(0) = u_0 \quad \text{on } \mathbb{R},$$

with

$$u_0(x) = \begin{cases} 1 & , x \le 0 \\ 1-x & , 0 \le x \le 1 \\ 0 & , x \ge 1. \end{cases}$$

We get, $0 \le t \le 1$,

$$u(t,x) = \begin{cases} 1 & , x \le t, \\ \frac{1-x}{1-t} & , t \le x \le 1, \\ 0 & , x \ge 1. \end{cases}$$



Definition

A map $u \in L^\infty([0,T] \times \mathbb{R})$ is a weak solution to

$$\begin{aligned} \partial_t u + \frac{1}{2}\partial_x u^2 &= 0 \quad \text{on } (0,T) \times \mathbb{R} \\ u(0) &= u_0 \quad \text{on } \mathbb{R} \end{aligned}$$

if for all $\varphi \in C_c^\infty([0,T) \times \mathbb{R})$

$$\int_0^T \int_{\mathbb{R}} u\partial_t\varphi + \frac{1}{2}u^2\partial_x\varphi \, dxdt + \int_{\mathbb{R}} u_0\varphi(0)\, dx = 0.$$

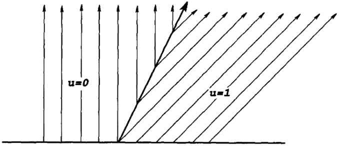
Consider the Burgers equation

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = u_0,$$

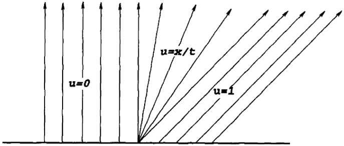
$$u_0(x) = \begin{cases} 0 & , x < 0 \\ 1 & , x \geq 0. \end{cases}$$

We set

$$u_1(t,x) = \begin{cases} 0 & , x < \frac{t}{2} \\ 1 & , x \geq \frac{t}{2} \end{cases}$$



$$u_2(t,x) = \begin{cases} 1 & , x > t, \\ \frac{x}{t} & , 0 < x < t \\ 0 & , x < 0. \end{cases}$$



Derivation of the entropy formulation: Let S be convex, C^2 and consider

$$\partial_t u^\varepsilon + \partial_x A(u^\varepsilon) = \varepsilon \partial_{xx} u^\varepsilon \quad \text{on } (0,T) \times \mathbb{R}.$$

Then

$$\begin{aligned}
\partial_t S(u^\varepsilon) &= -S'(u^\varepsilon)\partial_x A(u^\varepsilon) + \varepsilon S'(u^\varepsilon)\partial_{xx} u^\varepsilon \\
&= -S'(u^\varepsilon)A'(u^\varepsilon)\partial_x u^\varepsilon + \varepsilon S'(u^\varepsilon)\partial_{xx} u^\varepsilon \\
&= -\partial_x A^S(u^\varepsilon) + \varepsilon S'(u^\varepsilon)\partial_{xx} u^\varepsilon,
\end{aligned}$$

where $(A^S)' = S'A'$. We have

$$\begin{aligned}
\partial_{xx} S(u^\varepsilon) &= \partial_x (S'(u^\varepsilon)\partial_x u^\varepsilon) \\
&= S'(u^\varepsilon)\partial_{xx} u^\varepsilon + S''(u^\varepsilon)(\partial_x u^\varepsilon)^2.
\end{aligned}$$

Hence,

$$\partial_t S(u^\varepsilon) = -\partial_x A^S(u^\varepsilon) + \varepsilon \partial_{xx} S(u^\varepsilon) - \varepsilon S''(u^\varepsilon)(\partial_x u^\varepsilon)^2.$$

Assume $u^\varepsilon \to u$ pointwise. Then, in the sense of distributions,

$$\partial_t S(u) + \partial_x A^S(u) = -\lim_{\varepsilon \to 0} \varepsilon S''(u^\varepsilon)(\partial_x u^\varepsilon)^2 \le 0.$$

Definition

A function $u \in C([0,T];L^1(\mathbb{R}^d)) \cap L^\infty([0,T]\times\mathbb{R}^d)$ is an entropy solution if for all convex $S \in C^1(\mathbb{R})$ we have, in the sense of distributions,

$$\partial_t S(u) + \text{div} A^S(u) \leq 0 \quad \text{on } (0,T)\times\mathbb{R}^d$$

with $(A^S)' = S'A'$.

Remark

Equivalent formulation: For all $k \in \mathbb{R}$ consider the Kruzkov entropies $S(u) = |u-k|$. We obtain

$$\partial_t |u-k| + div([A(u)-A(k)]sgn(u-k)) \leq 0 \quad on\ (0,T)\times\mathbb{R}^d.$$

Derivation of the kinetic formulation: Let

$$\chi(u,v) = 1_{v<u} - 1_{v<0} = \begin{cases} +1 & , 0 \le v < u \\ -1 & , u \le v < 0 \\ 0 & , \text{ otherwise.} \end{cases}$$

Remark

Let $S' \in L^\infty_{loc}(\mathbb{R})$. Then, for all $u, u^1, u^2 \in \mathbb{R}$,

$$\int_{\mathbb{R}} S'(v)\chi(u,v)\,dv = S(u) - S(0)$$

$$\int_{\mathbb{R}} |\chi(u^1,v) - \chi(u^2,v)|\,dv = \int_{\mathbb{R}} |\chi(u^1,v) - \chi(u^2,v)|^2\,dv = |u^1 - u^2|$$

$$\partial_v \chi(u,v) = \delta_0 - \delta_{v=u}.$$

For given $v \in \mathbb{R}$ we choose $S_v(u) = |u-v| - |v|$ to get

$$\begin{aligned}
&\partial_t(|u-v|-|v|) + \sum_{i=1}^{d} \partial_{x_i}([A_i(u)-A_i(v)]\mathrm{sgn}(u-v) - [A_i(0)-A_i(v)]\mathrm{sgn}(v)) \\
&=: -2m(t,x,v) \\
&\leq 0.
\end{aligned}$$

Taking the derivative in v and dividing by -2 since

$$-\frac{1}{2}\partial_v(|u-v|-|v|) = \chi(v,u)$$

$$-\frac{1}{2}\partial_v([A_i(u)-A_i(v)]\mathrm{sgn}(u-v) - [A_i(0)-A_i(v)]\mathrm{sgn}(v)) = a_i(v)\chi(u,v)$$

we get (with $a := A'$)

$$\partial_t\chi + a(v)\cdot\nabla\chi = \partial_v m.$$

Definition

Let $u \in C([0,T];L^1(\mathbb{R}^d)) \cap L^\infty([0,T]\times\mathbb{R}^d)$. Then u is a kinetic solution if there is a non-negative, finite measure m such that $\chi(t,x,v) = \chi(u(t,x),v)$ satisfies

$$\partial_t\chi + a(v)\cdot\nabla\chi = \partial_v m.$$

Theorem

Let $u \in C([0,T];L^1(\mathbb{R}^d)) \cap L^\infty([0,T]\times\mathbb{R}^d)$. *Then* u *is an entropy solution iff* $\chi(t,x,v) = \chi(u(t,x),v)$ *is a kinetic solution.*

Definition

A function $f \in C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}))$ satisfying $f(t,x,v) = 0$ for all $|v| > R$ for some $R > 0$ is a generalized kinetic solution if

1. There is a non-negative, finite measure m such that

$$\begin{aligned}\partial_t f + a(v)\cdot\nabla f &= \partial_v m\\ f(0,x,v) &= \chi(u_0(x),v).\end{aligned}$$

2. There is a non-negative measure ν such that

$$\partial_v f(t,x,v) = \delta_{v=0} - \nu.$$

3. $|f(t,x,v)| = \operatorname{sgn}(v) f(t,x,v) \leq 1.$

Remark

1. *Entropy solutions are generalized entropy solutions. Indeed* $\partial_v \chi(t,x,v) = \delta_{v=0} - \delta_{v=u(t,x)}$.
2. *The set of generalized kinetic solutions is convex.*
3. *The set of generalized kinetic solutions is weakly closed: Let* f^n *be generalized kinetic solutions such that* $f^n \rightharpoonup f$, $m^n \rightharpoonup^* m$, $\nu^n \rightharpoonup^* \nu$ *with* $f \in C([0,T]; L^1(\mathbb{R}^d \times \mathbb{R}))$. *Then* f *is a generalized kinetic solution.*

Theorem

Let $u_0 \in L^1(\mathbb{R}^d)$ *and* f *be a generalized kinetic solution. Then there exists a* $u \in C([0,T]; L^1(\mathbb{R}^d))$ *such that* $f(t,x,v) = \chi(u(t,x),v)$ *a.e.. For two kinetic solutions* u^1, u^2 *with initial conditions* $u_0^1, u_0^2 \in L^1(\mathbb{R}^d)$ *we have*

$$\sup_{t\in[0,T]} \|u^1(t) - u^2(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0^1 - u_0^2\|_{L^1(\mathbb{R}^d)}.$$

Theorem

Let $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$. Then there exists a unique entropy solution. We have, for all $t \geq 0$,

$$\begin{aligned} \|u(t)\|_{L^p(\mathbb{R}^d)} &\leq \|u_0\|_{L^p(\mathbb{R}^d)} \quad \forall p \in [1,\infty] \\ \|u(t)\|_{BV(\mathbb{R}^d)} &\leq \|u_0\|_{BV(\mathbb{R}^d)} \end{aligned}$$

and

$$\begin{aligned} \int m \, dtdxdv &\leq \frac{1}{2}\|u_0\|_{L^2}^2, \\ \int m(t,x,v)\, dtdx &\leq \|u_0\|_{L^1}. \end{aligned}$$

Deterministic theory

Diffusion-Dispersion limits

- Reference [Hwang, Tzavaras; CPDE 2002]
- Have seen: Vanishing viscosity solutions correspond to entropy solutions.
- However: Other physically relevant approximations can lead to other types of solutions.
- We consider

$$\partial_t u^\varepsilon + \mathrm{div} A(u^\varepsilon) = \varepsilon \Delta u^\varepsilon + \delta \sum_{j=1}^{d} \partial_{x_j x_j x_j} u^\varepsilon \quad \text{on } (0,T)\times\mathbb{R}^d$$

with A Lipschitz.

- Kinetic form: For $\chi(t,x,v)=\chi(u(t,x),v)$ we get (1-d for simplicity)

$$
\begin{aligned}
\partial_t \chi^\varepsilon &= \delta_{u^\varepsilon=v}\partial_t u^\varepsilon \\
&= \delta_{u^\varepsilon=v}(-\partial_x A(u^\varepsilon)+\varepsilon\partial_{xx}u^\varepsilon+\delta\partial_{xxx}u^\varepsilon) \\
&= -A'(v)\partial_x\chi^\varepsilon+\delta_{u^\varepsilon=v}\partial_x(\varepsilon\partial_x u^\varepsilon+\delta\partial_{xx}u^\varepsilon)) \\
&= -a(v)\partial_x\chi^\varepsilon+\partial_x(\delta_{u^\varepsilon=v}(\varepsilon\partial_x u^\varepsilon+\delta\partial_{xx}u^\varepsilon)) \\
&-(\delta'_{u^\varepsilon=v}\partial_x u^\varepsilon(\varepsilon\partial_x u^\varepsilon+\delta\partial_{xx}u^\varepsilon)) \\
&= -a(v)\partial_x\chi^\varepsilon+\partial_x(\delta_{u^\varepsilon=v}(\varepsilon\partial_x u^\varepsilon+\delta\partial_{xx}u^\varepsilon)) \\
&+\partial_v(\delta_{u^\varepsilon=v}(\varepsilon(\partial_x u^\varepsilon)^2+\delta\partial_x u^\varepsilon\partial_{xx}u^\varepsilon)).
\end{aligned}
$$

- In general dimension:

$$\partial_t \chi^\varepsilon + a(v)\cdot\nabla\chi^\varepsilon = \sum_{j=1}^{d}\partial_{x_j}\left(\delta_{u^\varepsilon=v}(\varepsilon\partial_{x_j}u^\varepsilon + \delta\partial^2_{x_jx_j}u^\varepsilon)\right)$$
$$+\partial_v\left(\delta_{u^\varepsilon=v}(\varepsilon|\nabla u^\varepsilon|^2 + \delta\sum_{j=1}^{d}(\partial_{x_j}u^\varepsilon)(\partial^2_{x_jx_j}u^\varepsilon))\right)$$
$$=: \sum_{j=1}^{d}\partial_{x_j}\pi_j^\varepsilon + \partial_v m^\varepsilon.$$

  One may show $\pi_j^\varepsilon \to 0$ in distribution and m^ε are uniformly bounded measures. In addition $\chi^\varepsilon \to \chi$ in L^p_{loc}. Hence,

$$\partial_t\chi + a(v)\cdot\nabla\chi = \partial_v m \quad \text{on } (0,T)\times\mathbb{R}^d\times\mathbb{R}$$

  for some finite measure m.
- If $\delta = o(\varepsilon^2)$ then $m \geq 0$. If $\delta = O(\varepsilon^2)$ then m is not known to have a definite sign.

Relaxation approximation:

- Consider

$$\partial_t u^\varepsilon + \operatorname{div} v^\varepsilon = 0$$
$$\partial_t v_i^\varepsilon + A_i^2 \partial_{x_i} u^\varepsilon = -\frac{1}{\varepsilon}(v_i^\varepsilon - A_i(u^\varepsilon))$$

with A_i chosen constants such that

$$\sum_{i=1}^{d} \left(\frac{A_i'(u)}{A_i}\right)^2 < 1 \quad \text{for } i = 1, \ldots, d,\, u \in \mathbb{R}.$$

For $\chi^\varepsilon = \chi(u^\varepsilon(t,x), v)$ it can be shown

$$\partial_t \chi^\varepsilon + a(v) \cdot \nabla \chi^\varepsilon = \sum_{j=1}^{d} \left(\bar{g}_j^\varepsilon + \partial_v g_j^\varepsilon\right) + \partial_t \left(\bar{g}_0^\varepsilon + \partial_v g_0^\varepsilon\right) + \partial_v m^\varepsilon$$

with, $i = 0, \ldots, d$,

$$\bar{g}_i^\varepsilon, g_i^\varepsilon \to 0 \text{ in } L^2_{t,x,v}$$

and m^ε are uniformly bounded measures. Again $\chi^\varepsilon \to \chi$ satisfying

$$\partial_t \chi + a(v) \cdot \nabla \chi = \partial_v m \quad \text{on } (0,T) \times \mathbb{R}^d \times \mathbb{R}.$$

Definition (DeLellis, Otto, Westdickenberg, 2003)

A function $u \in C([0,T];L^1(\mathbb{R}^d)) \cap L^\infty([0,T]\times\mathbb{R}^d)$ is said to be a quasi-solution if $\chi(t,x,v) = \chi(u(t,x),v)$ satisfies

$$\partial_t \chi + a(v)\cdot\nabla\chi = \partial_v m \quad \text{on } (0,T)\times\mathbb{R}^d\times\mathbb{R}$$

for some (signed) measure m satisfying $m([0,T]\times\mathbb{R}^d\times B_R(0)) < \infty$ for all $R > 0$.

Theorem (DeLellis, Westdickenberg, 2003)

Let $\lambda > \frac{1}{3}$. Then there exists a quasi-solution u, such that u is a weak solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0, \quad \text{on } (0,T)\times\mathbb{R}$$
$$u(0) = u_0 \in L^\infty(\mathbb{R})$$

and

$$u \notin L^1([0,T];W^{\lambda,1}(\mathbb{R})).$$

Example (Jabin, Perthame; 2002)

Consider an entropy solution u to

$$\partial_t u + \frac{1}{2}\partial_x u^2 = f \quad \text{on } (0,T)\times\mathbb{R}$$

with $f = \frac{1}{N^2}\sum_{i=1}^N \sum_{j=1}^N \delta_{t=\frac{i}{N}}\delta_{x=\frac{j}{N}}$. Note $\|f\|_{\mathscr{M}} \le 1$. Then, for some $C>0$,

$$\|u\|_{L^1_t W^{\lambda,1}_x} \ge C N^{\lambda-\frac{1}{2}}.$$

Thus expect: For $f\in L^1$, in general

$$u \notin L^1([0,T];W^{\lambda,1}(\mathbb{R}))$$

for $\lambda > \frac{1}{2}$.

Motivation

Stochastic scalar conservation laws - Motivation

Motivation

- Stochastic mean field equations:

$$dX_t^i = \sigma^L \left(X_t^i, \frac{1}{L} \sum_{j=1}^{L} \delta_{X_t^j} \right) \circ dz_t \quad \text{in } \mathbb{R}^N$$

for $i = 1, \dots, L$.

- Then the empirical distribution

$$\pi_t^L := \frac{1}{L} \sum_{j=1}^{L} \delta_{X_t^j}$$

satisfies

$$d\pi_t^L + \operatorname{div}(\sigma^L(x, \pi_t^L)\pi_t^L \circ dz) = 0,$$

where

$$\operatorname{div}(\sigma^L(x, \pi_t^L)\pi_t^L \circ dz) = \sum_{i,j=1}^{d} \partial_{x_i}(\sigma_{i,j}^L(x, \pi_t^L)\pi_t^L \circ dz^j).$$

Motivation

Indeed:

$$d\varphi(X_t^i) = \nabla\varphi(X_t^i)\cdot\sigma^L(X_t^i, \sum_{j=1}^{L} X_t^j)\circ dz_t$$

Hence

$$d\frac{1}{L}\sum_{i=1}^{L}\varphi(X_t^i) = \frac{1}{L}\sum_{i=1}^{L}\nabla\varphi(X_t^i)\cdot\sigma^L(X_t^i, \sum_{j=1}^{L} X_t^j)\circ dz_t$$

and get

$$\begin{aligned} d(\varphi, \pi_t^L) &= (\nabla\varphi(\cdot)\cdot\sigma^L(\cdot, \pi_t^L)\circ dz_t, \pi_t^L) \\ &= -(\varphi, \mathrm{div}(\pi_t^L\sigma^L(\cdot, \pi_t^L)\circ dz_t)). \end{aligned}$$

Motivation

- Suppose that π_t^L converges weak* to some π_t and $\sigma^L \to \sigma$ for some $\sigma : \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$.
- We assume that σ^L becomes local in the limit, i.e. for $\pi \in L^1(\mathbb{R}^d)$, $\sigma(x,\pi) = \sigma(x,\pi(x))$ for a.e. $x \in \mathbb{R}^d$.
- Then m_t evolves according to

$$d\pi + \mathrm{div}(\underbrace{\sigma(x,\pi)\pi}_{=:A(x,\pi)} \circ dz) = 0 \quad \text{on } (0,T)\times\mathbb{R}^d.$$

Motivation

- We consider PDE driven by a 'rough' signal z of the type

$$du + \operatorname{div}(A(x,u) \circ dz) = 0 \quad \text{on } (0,T) \times \mathbb{R}^d.$$

If A is a diagonal matrix this becomes

$$du + \sum_{j=1}^{d} \partial_{x_j} A_j(x,u) \circ dz_j = 0 \quad \text{on } (0,T) \times \mathbb{R}^d.$$

Spatially homogeneous case

Well-posedness - Spatially homogeneous case

Spatially homogeneous case

- We start with the spatially homogeneous case and A being a diagonal matrix, that is:

$$du + \sum_{j=1}^{d} \partial_{x_j} A_j(u) \circ dz_j = 0 \quad \text{on } (0,T) \times \mathbb{R}^d. \tag{$\star$}$$

  Here z is assumed to be a continuous function ('rough' = continuous).
- If z is smooth, then $(\star)$ makes sense classically

$$du + \sum_{j=1}^{d} \partial_{x_j} A_j(u) \dot{z}_j = 0 \quad \text{on } (0,T) \times \mathbb{R}^d.$$

- Aims:
  - Intrinsic solution: Define solutions to $(\star)$ and prove well-posedness.
  - Consistency: Show that solutions to $(\star)$ are obtained by approximation of the driving signal z.

Spatially homogeneous case

- Kinetic form: We use

$$\chi(u,v) = 1_{v<u} - 1_{v<0} = \begin{cases} +1 & , 0 \le v < u \\ -1 & , u \le v < 0 \\ 0 & , \text{ otherwise} \end{cases}$$

  and set $\chi(t,x,v) := \chi(u(t,x),v)$. As in the deterministic case

$$\partial_t \chi(t,x,v) + \sum_{j=1}^{d} A_j'(v) \partial_{x_j} \chi(t,x,v) \dot{z}_j = \partial_v m(t,x,v). \qquad (\star)$$

- Advantage: $(\star)$ is a linear equation in χ, at the expense of introducing the additional velocity variable v.
- In contrast to the non-linear situation, $(\star)$ can be transformed in a 'robust' form, i.e. in a form making sense also for non-smooth z.
- Here we follow the principle idea of stochastic viscosity solutions, i.e. do not transform the PDE itself, but put the transformation into test-functions.

Spatially homogeneous case

- Choose the test-functions φ as solutions to the transport equation

$$\partial_t\varphi(t,x,v)+\sum_{j=1}^{d}A_j'(v)\partial_{x_j}\varphi(t,x,v)\dot{z}_j=0 \quad \text{on } (0,T)\times\mathbb{R}^d\times\mathbb{R}. \tag{1}$$

Spatially homogeneous case

- Then consider *convolutions along characteristics:*

$$
\begin{aligned}
\partial_t(\chi * \varphi) &= \partial_t \int \chi(t,x,v)\varphi(t,x,v)dx \\
&= \int \left(-\sum_{j=1}^{d} A_j'(v)\partial_{x_j}\chi(t,x,v)\dot{z}_j + \partial_v m \right) \varphi(t,x,v)dx \qquad (2)\\
&- \int \chi(t,x,v) \left(\sum_{j=1}^{d} A_j'(v)\partial_{x_j}\varphi(t,x,v)\dot{z}_j \right) dx \\
&= -\sum_{j=1}^{d} \int A_j'(v)\dot{z}_j \Big(\varphi(t,x,v)\partial_{x_j}\chi(t,x,v) + \chi(t,x,v)\partial_{x_j}\varphi(t,x,v) \Big) dx \\
&+ \int (\partial_v m)\,\varphi(t,x,v)dx = \int (\partial_v m)\varphi(t,x,v)dx.
\end{aligned}
$$

- The point is that φ in (1) is well-defined also for continuous z, thus (2) is well-defined for z continuous
  $\rightarrow$ use (2) as the a definition of a solution: *pathwise entropy solution*.

Spatially homogeneous case

- It remains to give meaning to

$$\partial_t\varphi(t,x,v)+\sum_{j=1}^{d}A_j'(v)\partial_{x_j}\varphi(t,x,v)\dot{z}_j=0 \quad \text{on } (0,T)\times\mathbb{R}^d\times\mathbb{R} \qquad (\star)$$

for continuous signals z.

- Method of characteristics for $(\star)$ gives

$$\varphi(t,x,v)=\varphi^0(x+A'(v)z(t)).$$

Definition

A function $u\in C([0,T];L^1(\mathbb{R}^d))\cap L^\infty([0,T]\times\mathbb{R}^d)$ is a pathwise entropy solution if there is a non-negative, finite measure m such that for all φ as above we have, in the sense of distributions,

$$\partial_t(\chi*\varphi)=\partial_t\int\chi(t,x,v)\varphi(t,x,v)dx=\int(\partial_v m)(t,x,v)\varphi(t,x,v)dx.$$

Spatially homogeneous case

Theorem (Lions, Perthame, Souganidis; SPDE 2013)

1. *For each* $u_0 \in (L^\infty \cap BV)(\mathbb{R}^d)$ *there is a pathwise entropy solution.*
2. *Let* $u^1, u^2 \in L^\infty([0,T]; BV(\mathbb{R}^d))$ *be two pathwise entropy solutions with driving signals* $z^1, z^2 \in C_0([0,T]; \mathbb{R}^d)$. *Then,*

$$\|u^1(t) - u^2(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0^1 - u_0^2\|_{L^1(\mathbb{R}^N)} + C\sqrt{\|z^1 - z^2\|_{C([0,t];\mathbb{R}^d)}}.$$

- In particular, this yields consistency.

Spatially homogeneous case

Comments on the proof:

- Want to estimate

$$\partial_t \int |u^1 - u^2| dy = \partial_t \int |\chi^1 - \chi^2|^2 dydv.$$

- To use the definition, estimate instead

$$\partial_t \int |\chi^1 * \varphi^\varepsilon - \chi^2 * \varphi^\varepsilon|^2 dydv$$

  with φ^ε test-functions transported along characteristics.
- *Doubling the variables*: One considers a family of testfunctions

$$\varphi^\varepsilon(t,x,y,v) = \varphi^{0,\varepsilon}(y - x + A'(v)z(t)) \underset{\varepsilon\to 0}{\longrightarrow} \delta(y - x + A'(v)z(t)).$$

Spatially homogeneous case

- Leads to error terms, due to doubling of the variables, that need to be controlled. An important one:

$$\begin{aligned}
&\partial_v \varphi^\varepsilon(x,y,v,t)\varphi^\varepsilon(x,y',v,t) + \varphi^\varepsilon(x,y,v,t)\partial_v \varphi^\varepsilon(x,y',v,t) \\
&= (D\varphi^\varepsilon)(x,y,v,t)A''(v)z(t)\varphi^\varepsilon(x,y',v,t) + \varphi^\varepsilon(x,y,v,t)(D\varphi^\varepsilon)(x,y',v,t)A''(v)z(t) \\
&= \partial_y \varphi^\varepsilon(x,y,v,t)A''(v)z(t)\varphi^\varepsilon(x,y',v,t) + \varphi^\varepsilon(x,y,v,t)\partial_y \varphi^\varepsilon(x,y',v,t)A''(v)z(t) \\
&= \partial_y(\varphi^\varepsilon(x,y,v,t)\varphi^\varepsilon(x,y',v,t))A''(v)z(t),
\end{aligned}$$

which vanishes after an integration in y
$\rightarrow$ crucial cancellation which uses the simple structure of the characteristics.

Spatially inhomogeneous case

Well-posedness - Spatially inhomogeneous case

Spatially inhomogeneous case

- Let us now consider the spatially inhomogeneous case:

$$du + \sum_{j=1}^{d} \partial_{x_j} A_j(x,u) \circ dz_j = 0 \quad \text{on } (0,T) \times \mathbb{R}^d.$$

- The principle idea remains the same: We pass to the kinetic formulation:

$$d\chi + \sum_{j=1}^{d} (\partial_u A_j)(x,v)\partial_{x_j}\chi \dot{z}_j - \left(\sum_{j=1}^{d} (\partial_{x_j} A_j)(x,v)\dot{z}_j\right) \partial_v \chi = \partial_v m.$$

- Again we test by solutions to

$$\partial_t \varphi + \sum_{j=1}^{d} (\partial_u A_j)(x,v)\partial_{x_j}\varphi \dot{z}_j - \left(\sum_{j=1}^{d} (\partial_{x_j} A_j)(x,v)\dot{z}_j\right) \partial_v \varphi = 0.$$

- As before one gets: *convolution along characteristics*

$$\partial_t(\chi * \varphi) = \int \partial_v m \varphi(t,x,v) dx dv.$$

Spatially inhomogeneous case

- The difficulty lies in the characteristics to

$$\partial_t \varphi + \sum_{j=1}^{d} (\partial_u A_j)(x,v)\partial_{x_j}\varphi \dot{z}_j - \left(\sum_{j=1}^{d} (\partial_{x_j} A_j)(x,v)\dot{z}_j\right) \partial_v \varphi = 0.$$

- In contrast to the spatially homogeneous case, the characteristics do not have an explicit solution anymore. Instead they are given as the solutions to the rough DE: (with $z^{t_1,j}(t) := z^j(t_1 - t)$.)

$$dX^i_{(t_1,x,v)}(t) = \sum_{j=1}^{d} (\partial_u A_j)(X_{(t_1,x,v)}(t), \Xi_{(t_1,x,v)}(t))\dot{z}^{t_1,j}(t),$$

$$d\Xi_{(t_1,x,v)}(t) = -\sum_{j=1}^{d} (\partial_{x_j} A_j)(X_{(t_1,x,v)}(t), \Xi_{(t_1,x,v)}(t))\dot{z}^{t_1,j}(t),$$

$$X^i_{(t_1,x,v)}(0) = x^i \text{ and } \Xi_{(t_1,x,v)}(0) = v.$$

- We get

$$\varphi_{t_0}(t,x,v) = \varphi^0 \begin{pmatrix} X_{(t,x,v)}(t - t_0) \\ \Xi_{(t,x,v)}(t - t_0) \end{pmatrix}.$$

Spatially inhomogeneous case

- Hence, to get well-posedness of φ we need stability of (X, Ξ) with respect to the driving signal z. I.e. rough path stability.
  $\rightarrow$ need z to be a rough path.

Theorem (Gess, Souganidis; CMS, 2015)

1. *For each $u_0 \in (L^1 \cap L^2)(\mathbb{R}^d)$ there is a pathwise entropy solution.*
2. *Let $u^1, u^2 \in L^\infty([0,T]; L^1(\mathbb{R}^d))$ be two pathwise entropy solutions with the same driving signal z. Then,*

$$\text{ess sup}_{t\in[0,T]} \|u^1(t) - u^2(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0^1 - u_0^2\|_{L^1(\mathbb{R}^d)}.$$

Spatially inhomogeneous case

Comments on the proof:

- One looses the cancellation effect from the homogeneous case.
- Instead, the error has to be carefully controlled.
- Key new step: Interval splitting + rough path estimates for the characteristics
- Drawback: Do not get a quantitative continuous dependence on the driving rough paths anymore.