Part 2: Well-posedness by noise for scalar conservation laws

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- Classical well-posedness for ODE:

$$\begin{aligned} dX_t^x &= b(X_t^x)dt \\ X_0^x &= x \end{aligned}$$

is well-posed if b is sufficiently smooth, e.g. Lipschitz continuous.

- In contrast, well-posedness for SDE: $(\sigma > 0)$

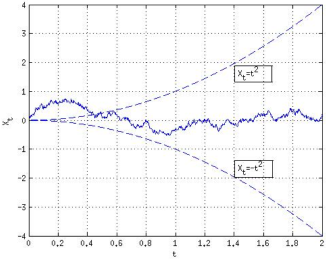
$$\begin{aligned} dX_t^x &= b(X_t^x)dt + \sigma d\beta_t \\ X_0^x &= x \end{aligned}$$

has a unique solution if b is bounded, measurable. This is called '*well-posedness by noise*'.

- A simple example: Consider

$$dX_t^x = b(X_t^x)dt + \sigma d\beta_t$$
$$X_0^x = x$$

with $b(x) = 2\mathrm{sgn}(x)\sqrt{|x|}$:



- One reason: Fokker-Planck equation for the law $u(t,\cdot) = \mathscr{L}(X_t^x)(\cdot)$

$$\partial_t u = \frac{\sigma^2}{2}\Delta u - \mathrm{div}(bu).$$

- Again consider

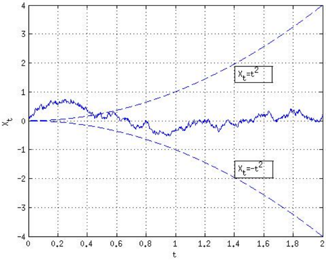
$$dX_t^x = b(X_t^x)dt + \sigma d\beta_t$$
$$X_0^x = x,$$

with $b(x) = 2\text{sgn}(x)\sqrt{|x|}$.

- We may select solutions among the set of solutions to the non-perturbed problem by considering the zero noise limit $\sigma \to 0$.
- One can show e.g. [Bafico, Baldi; 1982], [Flandoli, Delarue; 2013]

$$\mathscr{L}(X_\cdot^0) \rightharpoonup \frac{1}{2}\delta_{x^+(\cdot)} + \frac{1}{2}\delta_{x^-(\cdot)}$$

with $x^\pm$ the extremal solutions.



- Key hope in SPDE: Establish similar effects for PDE, in particular in fluid dynamics, e.g. $3d$-Navier-Stokes equations.
- Problem: A-priori unclear which form of noise to consider
- Additive noise, e.g.

$$du = \Delta u\,dt + f(u)\,dt + dW_t \quad \text{[Gyöngy, Pardoux; 1993]}$$

$$du + (u\cdot\nabla)u\,dt + \nabla p\,dt = \Delta u\,dt + dW_t \quad \text{[Flandoli, Romito; 2007]}.$$

- Formally, the Fokker-Planck equation becomes an infinite dimensional PDE.
- Many attempts to uniqueness, infinite dimensional analysis: Albeverio, Bogachev, Cerrai, DaPrato, Flandoli, Röckner, Romito...
- However, uniqueness for the stochastic $3d$-Navier-Stokes equation remains open.
- More recent: Linear multiplicative noise

$$du + b(x)\cdot\nabla u\,dt = \nabla u \circ d\beta_t. \quad \text{[Flandoli, Gubinelli, Priola; Invent. Math., 2010]}.$$

- We recall: Consider
$$\partial_t u + b(x) \cdot \nabla u = 0, \tag{TE}$$
for non-Lipschitz b (but, say, Hölder continuous). E.g. $b(x) = 2\text{sgn}(x)\sqrt{|x|}$.
- Characteristics for (TE):
$$dX_t^x = b(X_t^x)dt \in \mathbb{R}^d$$
$$X_0^x = x.$$
- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than $u(t) \in BV$ even if u_0 is smooth.
- Characteristics branch causing non-uniqueness of weak solutions.
- Question: Can noise restore uniqueness or increase regularity?

- Consider
$$du + b(x) \cdot \nabla u \, dt = \sigma \nabla u \circ d\beta_t. \tag{STE}$$
- Characteristics for (STE):
$$\begin{aligned} dX_t^x &= b(X_t^x)dt - \sigma d\beta_t \in \mathbb{R}^d \\ X_0^x &= x. \end{aligned}$$
- Two (related) effects: regularization by noise, well-posedness by noise.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; 2010]: Weak solutions to (STE) are unique.
- Regularization by noise [Flandoli, Fedrizzi; 2013]: If u_0 is smooth then $u(t)$ is smooth.

- As for ODE this may be used to obtain selection principles for the deterministic case.
- Again consider

$$du + b(x) \cdot \nabla u = 0$$
$$u(0) = 1_{[0,\infty)}$$

with $b(x) = 2\text{sgn}(x)\sqrt{|x|}$.

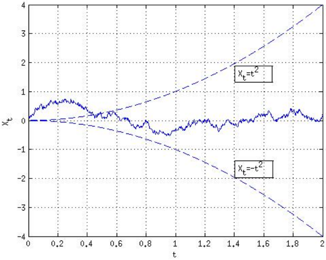
- Then there are multiple weak solutions.
- Vanishing viscosity :

$$du^\varepsilon + b(x) \cdot \nabla u^\varepsilon = \varepsilon \Delta u^\varepsilon$$

- Then:

$$u^\varepsilon \to u = \frac{1}{2}u_1 + \frac{1}{2}u_2$$
$$= 1_{\{x \geq x^+\}} + \frac{1}{2}1_{\{x^- < x < x^+\}},$$

with $u_1 = 1_{\{x \geq x^+\}}$, $u_2 = 1_{\{x > x^-\}}$.



- Zero noise limit [Attanasio, Flandoli; 2009]:

$$du^{\sigma} + b(x) \cdot \nabla u^{\sigma} = \sigma \nabla u^{\sigma} \circ d\beta_t.$$

Then:

$$\mathscr{L}(u^{\sigma}) \to \frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2}$$

with $u_1 = 1_{\{x \geq x^+\}}$, $u_2 = 1_{\{x > x^-\}}$.

- Left open: What about the nonlinear case, e.g. Burgers?
- Linear multiplicative noise does not help anymore:

$$du + \partial_x u^2\, dt = \partial_x u \circ d\beta_t.$$

Then: $v(t,x) := u(t, x-\beta_t)$ is a solution to

$$\partial_t v + \partial_x v^2 = 0.$$

- In particular, weak solutions are non-unique.
- Conclusion in [Flandoli, Gubinelli, Priola; *Invent. Math.*, 2010]:
  „It is very easy to produce examples [...] for a stochastic version of Euler equation which show that the particular noise we use does not have any regularizing effect in this case. [...] The generalization to nonlinear transport equations, where b depends on u itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem.“

Well-posedness by noise for stochastic scalar conservation laws

Well-posedness by noise for stochastic scalar conservation laws

- Consider
$$\partial_t u + b(x,u)\cdot\nabla u = 0.$$
- Scalar conservation laws with irregular flux: Traffic flows, sedimentation processes
[De Philippis et al., *CPDE*, 2015; Andreianov, Karlsen, Risebro, *ARMA*, 2011].
- In this talk: For simplicity consider
$$\partial_t u + b(x)\cdot\nabla(u^2) = 0,$$
for irregular b (in particular $\mathrm{div} b \notin L^\infty$).
- The deterministic problem is ill-posed in general: Entropy solutions are non-unique.

- Model example: Consider

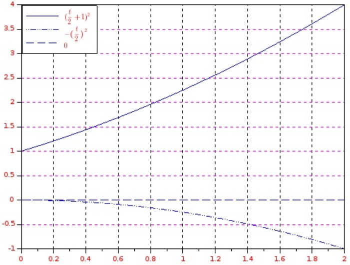
$$\begin{aligned} \partial_t u + b(x) \cdot \nabla(u^2) &= 0 \\ u(0,x) &= 1_{[0,1]}(x) \end{aligned} \qquad (\star)$$

with $b(x) = 2\mathrm{sgn}(x)(\sqrt{|x|} \wedge K)$.

- Rankine-Hugoniot implies:

$$u^1(t,x) := \begin{cases} 1 & \text{if } 0 \le x \le \left(\frac{t}{2}+1\right)^2 \\ 0 & \text{otherwise,} \end{cases}$$

$$u^2(t,x) := \begin{cases} 1 & \text{if } -\left(\frac{t}{2}\right)^2 \le x \le \left(\frac{t}{2}+1\right)^2 \\ 0 & \text{otherwise.} \end{cases}$$



- Can we restore well-posedness by adding a linear multiplicative noise term?
- Non-trivial: shocks due to the nonlinearity and shocks due to the irregularity of b may combine in such a way that this noise may be insufficient.
- Stochastic Burgers' equation:

$$du + b(x) \cdot \nabla(u^2)dt = \nabla u \circ d\beta_t.$$

Theorem

Assume $b \in (W^{1,1} \cap L^\infty)(\mathbb{R}^d)$ and $\operatorname{div} b \in L^p(\mathbb{R}^d)$ for some $p > d$. For $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$, the stochastic Burgers' equation admits a unique entropy solution.

Model example: $b(x) = 2\operatorname{sgn}(x)(\sqrt{|x|} \wedge K)$.

- Given $u = u(t, \omega, x)$, introduce a new (velocity) variable $v \in \mathbb{R}$ and define the kinetic function:

$$f = f[u](t, \omega, x, v) = 1_{v<u(t,x)} - 1_{v<0}.$$

- Informal computation (pretending solution is smooth):

$$\begin{aligned}
\partial_t f &= \partial_t 1_{v<u(t,x)} \\
&= \delta_{v=u(t,x)} \partial_t u(t,x) \\
&= \delta_{v=u(t,x)} (-b(x) \cdot \nabla(u^2) dt + \nabla u \circ d\beta_t) \\
&= \delta_{v=u(t,x)} (-2b(x) \cdot u \nabla u dt + \nabla u \circ d\beta_t) \\
&= -2b(x) \cdot v \delta_{v=u(t,x)} \nabla u dt + \delta_{v=u(t,x)} \nabla u \circ d\beta_t \\
&= -2b(x) \cdot v \nabla f dt + \nabla f \circ d\beta_t.
\end{aligned}$$

  Hence,

$$\partial_t f + 2b(x) v \cdot \nabla_x f - \nabla_x f \circ d\beta_t = 0.$$

- But: Need to take into account shocks and entropy dissipation.

Definition

u is an entropy solution if $f[u]$ is adapted and solves (in the sense of distributions) the kinetic equation

$$\partial_t f + 2b(x)v \cdot \nabla_x f - \nabla_x f \circ d\beta_t = \partial_v m$$

for some nonnegative random measure m on $[0,T] \times \mathbb{R}^d_x \times \mathbb{R}_v$.

Existence of a generalized entropy solution:

Approximate b by smooth b^ε. Consider

$$du^\varepsilon + b^\varepsilon(x)\cdot\nabla(u^\varepsilon)^2 dt = \nabla u^\varepsilon \circ d\beta_t$$

and its kinetic form

$$f^\varepsilon = f[u^\varepsilon](t,\omega,x,v) = 1_{v<u^\varepsilon(t,x)} - 1_{v<0} \qquad (\star)$$

solving

$$\partial_t f^\varepsilon + 2b^\varepsilon(x)v\cdot\nabla_x f^\varepsilon - \nabla_x f^\varepsilon \circ d\beta_t = \partial_v m^\varepsilon.$$

Proposition

We have the uniform bounds

$$\text{ess sup}_{\omega\in\Omega} \sup_{t\in[0,T]} \|u^\varepsilon(t)\|_{L^\infty} \le \|u_0\|_\infty$$

and for all $p\in[1,\infty)$

$$\text{ess sup}_{\omega\in\Omega} \sup_{t\in[0,T]} \|u^\varepsilon(t)\|_{L^p_x}^p + \|m^\varepsilon\|_{\mathscr{M}^1_{t,x,\xi}} \le C = C(\|divb\|_{L^1}) < \infty$$

- Consider the transformation $v^\varepsilon(t,x) := u^\varepsilon(t,x+\beta_t)$. Then v^ε solves

$$dv^\varepsilon + b^\varepsilon(x+\beta_t)\cdot\nabla(v^\varepsilon)^2 dt = 0.$$

The maximum principle yields

$$\sup_{t\in[0,T]} \|v^\varepsilon(t)\|_{L^\infty} \le \|u_0\|_\infty.$$

- Multiply kinetic form for v^ε by $v^{[p-1]}$ and integrate to get

$$\begin{aligned}\partial_t \int \frac{|v^\varepsilon|^p}{p}dx + \int 2b^\varepsilon(x+\beta_t)v^{[p-1]}\cdot\nabla_x f^\varepsilon dxdv &= \int v^{[p-1]}\partial_v m^\varepsilon dxdv \\ &= -(p-1)\int |v|^{p-2} m^\varepsilon dxdv.\end{aligned}$$

Hence

$$\partial_t \int \frac{|v^\varepsilon|^p}{p}dx - \underbrace{\int 2\mathrm{div} b^\varepsilon(x+\beta_t)\frac{|v^\varepsilon|^p}{p}dx}_{\le C\|\mathrm{div} b^\varepsilon\|_{L^1}\|v^\varepsilon\|_{L^\infty}^p} = -(p-1)\int |v|^{p-2} m^\varepsilon dxdv.$$

- May extract weak(*)-convergent subsequences of $f^\varepsilon \rightharpoonup f$, $m^\varepsilon \rightharpoonup^* m$ and pass to the limit in

$$\partial_t f^\varepsilon + 2b^\varepsilon(x)v \cdot \nabla_x f^\varepsilon - \nabla_x f^\varepsilon \circ d\beta_t = \partial_v m^\varepsilon.$$

- Problem:

$$f^\varepsilon = f[u^\varepsilon](t,\omega,x,v) = 1_{v<u^\varepsilon(t,x)} - 1_{v<0}$$

is not preserved under weak limits.

Definition

$f = f(t,\omega,x,v)$ is a generalized entropy solution if f solves the kinetic equation

$$\partial_t f + 2b(x)v \cdot \nabla_x f - \nabla_x f \circ d\beta_t = \partial_v m$$

and, for some nonnegative measure ν,

$$|f| = \mathrm{sgn}(v) f \le 1, \quad \partial_v f = \delta_0 - \nu.$$

Proposition

Let $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ and b, $divb \in L^1(\mathbb{R}^d)$. Then there exists a generalized entropy solution.

Uniqueness of generalized entropy solutions:

- Main difficulty: Prove that every generalized entropy solution is an entropy solution.
- Given a generalized entropy solution f, we need to find a function u such that

$$f = 1_{v<u(t,x)} - 1_{v<0}.$$

- To do so it is enough to show that

$$|f| \in \{0,1\} \quad \text{a.e.}.$$

  Equivalently,

$$|f| - f^2 = 0 \quad \text{a.e.}.$$

  Indeed, $\partial_v f = \delta_0 - \nu$ shows that $v \mapsto f(t,x,v)$ is non-increasing on $\mathbb{R} \setminus \{0\}$ and $f(t,x,0+) - f(t,x,0-) = 1$.
- Since $|f| - f^2 \geq 0$ it only remains to obtain an upper estimate, i.e.

$$|f| - f^2 \leq 0 \quad \text{a.e.}.$$

- Steps of the proof:
  - First step (deterministic): Via renormalization arguments derive an inequality for $|f| - f^2$ (similar to the kinetic equation).
  - Second step (stochastic): Take the expectation and use *parabolic* theory.
- Kinetic equation

$$\partial_t f + 2b(x)v \cdot \nabla_x f - \nabla_x f \circ d\beta_t = \partial_v m$$

rewritten in Itô form:

$$\partial_t f + 2b(x)v \cdot \nabla_x f - \nabla_x f \, d\beta_t - \frac{1}{2}\Delta_x f = \partial_v m.$$

A Laplacian appears, which suggests regularization. Note: the equation is hyperbolic (not parabolic: no regularization of initial datum).

- Taking the expectation yields

$$\partial_t \mathbb{E}f + 2b(x)v \cdot \nabla_x \mathbb{E}f - \frac{1}{2}\Delta_x \mathbb{E}f = \partial_v \mathbb{E}m,$$

i.e. a parabolic PDE.

Renormalization step:

- By informal computations, $|f|-f^2$ satisfies

$$\partial_t(|f|-f^2)+2b(x)v\cdot\nabla_x(|f|-f^2)-\nabla_x(|f|-f^2)\circ d\beta_t=(\mathrm{sgn}(v)-2f)\partial_v m$$

- Rigorous: Need $b\in W^{1,1}$ for commutator estimates [DiPerna-Lions 89, Ambrosio 04].
- To deal with the right hand side we have to take the velocity average

$$\partial_t\int_v(|f|-f^2)+2b(x)\int_v v\cdot\nabla_x(|f|-f^2)-\int_v\nabla_x(|f|-f^2)\circ d\beta_t=\int_v(\mathrm{sgn}(v)-2f)\partial_v m$$

- Using $\partial_v f=\delta_0-v$, $\partial_v(\mathrm{sgn}(v))=2\delta_0$ and integration by parts for φ independent of v, we get

$$\int_v(\mathrm{sgn}(v)-2f)\partial_v m=-2\int_v vm\varphi\le 0$$

Thus,

$$\partial_t\int_v(|f|-f^2)+2b(x)\int_v v\cdot\nabla_x(|f|-f^2)-\int_v\nabla_x(|f|-f^2)\circ d\beta_t\le 0.$$

- Passing to Itô form and taking expectation

$$\partial_t \int_v \mathbb{E}(|f|-f^2) + 2b(x)\int_v v\cdot\nabla_x \mathbb{E}(|f|-f^2) - \Delta \int_v \mathbb{E}(|f|-f^2) \le 0.$$

But this is **not** a closed equation for $\int_v \mathbb{E}(|f|-f^2)$.

Corollary

With the previous assumptions, for testfunctions $\varphi = \varphi(t,x)$,

$$\partial_t(\mathbb{E}[|f| - f^2], \varphi) \leq (\mathbb{E}[|f| - f^2], \partial_t\varphi - 2div_x(b(x)v\varphi) + \frac{1}{2}\Delta_x\varphi).$$

Since we work with bounded solutions $u \in L^\infty([0,T] \times \mathbb{R}^d)$ it is sufficient to consider velocities $v \in [-R, R]$ with $R = \|u\|_\infty$.

Proposition

Fix $T > 0$ *and assume* $b \in L^\infty(\mathbb{R}^d)$, *divb in* $L^p(\mathbb{R}^d)$ *for some* $p > d$. *Then, there exists a* $\varphi \geq 0$, *independent of* v, *with* $\varphi_T \sim 1$, *such that*

$$\partial_t\varphi - 2div_x(b(x)v\varphi) + \frac{1}{2}\Delta_x\varphi \leq C$$

for some $C > 0$ *(independent of* T*).*

Then Gronwall's inequality yields $|f| - f^2 \leq 0$.

Theorem

Assume $b \in (W^{1,1} \cap L^\infty)(\mathbb{R}^d)$ *with* $\operatorname{div} b \in L^p$ *for some* $p > d$*. Then*

1. *Every generalized entropy solution is an entropy solution.*
2. *For* $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ *there is an entropy solution.*

Proof of the proposition in two steps. We aim at

$$\partial_t \varphi - 2\mathrm{div}_x(b(x)v\varphi) + \frac{1}{2}\Delta_x \varphi \leq C.$$

Step 1: Take $\varphi \geq 0$ a solution to

$$\partial_t \varphi + 2R|\mathrm{div}_x b|\varphi + \frac{1}{2}\Delta_x \varphi = 0.$$

Lemma

Assume div b in L^p for some $p > d$. Then φ is in $W^{1,\infty}(\mathbb{R}^d)$ (uniformly in time).

Proof of the lemma based on heat kernel estimates (here forward equation):

$$\varphi_t = P_t \varphi_0 + \int_0^t P_{t-s}(2R|\mathrm{div} b|\varphi_s)ds.$$

Control of $\|\nabla \varphi\|_{L^\infty_x}$ needed for step 2.

Step 2: Conclusion:

$$
\begin{aligned}
&\partial_t \varphi - 2\mathrm{div}(bv\varphi) + \frac{1}{2}\Delta\varphi \\
&= \partial_t \varphi - 2v\mathrm{div}(b)\varphi - 2vb\cdot\nabla\varphi + \frac{1}{2}\Delta\varphi \\
&\leq (\partial_t\varphi + 2R|\mathrm{div} b|\varphi + \frac{1}{2}\Delta\varphi) \\
&\quad + 2(-R|\mathrm{div} b| - v\mathrm{div} b)\varphi \\
&\quad - 2vb\cdot\nabla\varphi \\
&\leq \|b\|_{L^\infty}\|\nabla\varphi\|_{L^\infty} =: C
\end{aligned}
$$

Theorem

Entropy solutions are unique

- Take two entropy solutions u^1, u^2 with kinetic functions

$$f^i = 1_{v<u^i(t,x)} - 1_{v<0}.$$

- Then $f := \frac{1}{2}f^1 + \frac{1}{2}f^2$ is a generalized entropy solution.
- Hence, f is an entropy solution, that is,

$$\begin{aligned} 1_{v<u(t,x)} - 1_{v<0} =& f = \frac{1}{2}f^1 + \frac{1}{2}f^2 \\ =& \frac{1}{2}(1_{v<u^1(t,x)} - 1_{v<0}) + \frac{1}{2}(1_{v<u^2(t,x)} - 1_{v<0}) \\ =& \frac{1}{2}1_{v<u^1(t,x)} + \frac{1}{2}1_{v<u^2(t,x)} - 1_{v<0}. \end{aligned}$$

- Thus,

$$1_{v<u(t,x)} = \frac{1}{2}1_{v<u^1(t,x)} + \frac{1}{2}1_{v<u^2(t,x)}.$$