Part 3: Regularization by noise for SCL

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Introduction

- Recall: In nonlinear PDE, linear multiplicative noise may not help anymore

$$du + \frac{1}{2}\partial_x u^2 \, dt = \partial_x u \circ d\beta_t.$$

- Then: $v(t,x) := u(t, x - \beta_t)$ is the unique solution to

$$\partial_t v + \frac{1}{2}\partial_x v^2 = 0.$$

- Last part: Noise still helps regarding irregularities of spatial inhomogeneities, but not regarding nonlinear singularities.

Regularity of solutions to stochastic SCL

- First part: Stochastic scalar conservation laws

$$du+\sum_{j=1}^{d}\partial_{x_j}A_j(u)\circ d\beta_j=0, \tag{SSCL}$$

  on the torus $\mathbb{T}^d$, $A\in C^2$.

- Assume that the flux A is nonlinear: I.e. there exist $\theta\in(0,1]$ and $C>0$ such that, for all $\sigma\in S^{d-1}$, $z\in\mathbb{R}^d$ and $\varepsilon>0$,

$$|\{v\in\mathbb{R}:|A'(v)\sigma-z|\le\varepsilon\}|\le C\varepsilon^{\theta}.$$

- e.g. $A(u)=\frac{u^{l+1}}{l+1}$, then $\theta=\frac{1}{l}$, $l\ge 1$.

Theorem (G., Souganidis; CPAM, 2016)

Let $u \in L^\infty$ be a bounded quasi-solution to

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T}. \tag{SB}$$

Then,

$$u \in L^1_t W^{\lambda,1}_x \quad \text{for all } \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.}.$$

If u is an entropy solution, then

$$u(t) \in W^{\lambda,1}_x \quad \text{for all } t > 0, \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.}. \tag{$\star$}$$

Two resulting questions:

1. Can the zero set in $(\star)$ be chosen uniformly in t?
2. Characterize the properties of Brownian paths leading to $(\star)$.

Regularization by nonlinear noise

- Consider, for $w \in C([0,T])$,

$$du + \frac{1}{2}\partial_x u^2 \circ dw_t = 0, \quad \text{on } \mathbb{R}.$$

- Get

$$\|u(t)\|_{W_x^{1,\infty}} \leq \Big(\max_{0\leq s\leq t}(w(s)-w(t)) \wedge (w(t) - \min_{0\leq s\leq t} w(s)) \Big)^{-1}.$$

- Decisive path property: "Changing sign of the derivative".
- For $w = \beta$ we get

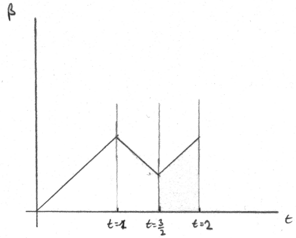
$$v(t) \in W^{1,\infty}, \quad \mathbb{P}-a.s.$$

- But: Zero set depends on time $t > 0$.

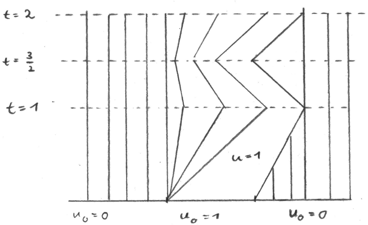
Regularization by nonlinear noise

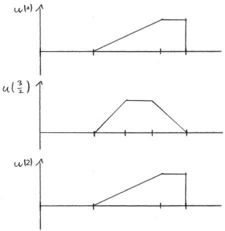
- Example:

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta = 0$$
$$u(0) = 1_{[0,1]}$$



- Solution u:





Path-by-path regularization by noise

Path-by-path regularization by noise

Framework

- Model example:
$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{T},$$
with $w \in C([0,T];\mathbb{R})$.
- General class:
$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} A^j(u) \circ dw_t^j = 0 \quad \text{on } \mathbb{T}^d,$$
with $A \in C^2(\mathbb{R},\mathbb{R}^d)$ is supposed to satisfy a nonlinearity condition: For some for some $\nu \geq 1$,
$$\inf_{e=(e_1,\dots,e_d))\in\mathbb{R}^d} \max_{i=1,\dots,d} |e_i(a^i(v_2)-a^i(v_1))| \geq |v_2-v_1|^\nu \quad \forall v_1, v_2 \in \mathbb{R}.$$
- How to classify irregularity properties of w?

Idea of the proof

- Ideas of the proof of regularity for

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T}.$$

- By definition quasi-solutions satisfy

$$d\chi + v\partial_x \chi \circ d\beta_t = \partial_v m,$$

  for some finite Radon measure m.

- Change of variables gives

$$\chi(t,x,v) = \chi_0(x + v\beta_t, v) + \int_0^t \partial_v m(s, x + v(\beta_t - \beta_s), v)ds.$$

- Averaging over velocity

$$u(t,x) = \int_v \chi = \int_v \chi_0(x + v\beta_t, v)dv + \int_0^t \int_v \partial_v m(s, x + v(\beta_t - \beta_s), v)dvds.$$

Framework

- Fourier transform in spatial variable

$$\hat{u}(t,n) = \int_V e^{-iv\beta_t n}\hat{\chi}_0(n,v)dv + \int_0^t \int_V e^{-iv(\beta_t-\beta_s)n}\partial_v \hat{m}(s,n,v)dvds.$$

- The oscillatory integrals have a regularizing effect, both in v and in $\beta_t - \beta_s$.
- For SDE this has been considered by [Catellier, Gubinelli; *SPA*, 2016]: A path $w \in C(\mathbb{R}_+;\mathbb{R}^d)$ is said to be (ρ,γ)-irregular if

$$|\int_s^t e^{i\langle a,w_r\rangle}dr| \lesssim (1+|a|)^{-\rho}|t-s|^{\gamma} \quad \forall a \in \mathbb{R}^d, s<t.$$

- Note:

$$\int_s^t e^{i\langle a,w_r\rangle}dr = \int_{\mathbb{R}} e^{i\langle a,x\rangle}dL_w^{s,t}(x) = \hat{L_w^{s,t}}(a)$$

the Fourier transform of the local time.

Main result

Theorem

Let $w \in C^{\eta}([0,T],\mathbb{R}^d)$ for some $\eta > 0$ be (ρ,γ)-irregular, u a bounded quasi-solution solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{T}.$$

Then, for all

$$\lambda < \frac{\rho(\eta+1)-(1-\gamma)}{(\rho\vee 1)(\eta+1)+(1-\gamma)},$$

we have

$$\|u\|_{L^1_t W^{\lambda,1}_x} < \infty.$$

Corollary

Let β^H be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}]$ and u be a bounded quasi-solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta_t^H = 0 \quad on\ \mathbb{T}. \tag{1}$$

Then, for all $\lambda < \frac{1}{1+2H}$,

$$\|u\|_{L_t^1 W_x^{\lambda,1}} < \infty.$$

- Note: Fully recover the probabilistic result from [G., Souganidis; *CPAM*, 2016]: For $H = \frac{1}{2}$ get $\lambda < \frac{1}{2}$.

A path-by-path scaling condition

A path-by-path scaling condition

Discussion of the path classification

- The proof given in [G., Souganidis; *CPAM*, 2016] uses the *scaling property* of Brownian motion and independence of increments.
- However: (ρ,γ)-irregularity depends on two parameters, also encoding a time regularity. Hence, does not seem to be optimal.
- Moreover: (ρ,γ)-irregularity not easy to check.
- To avoid the use of oscillatory integrals: Completely avoid Fourier methods in the proof (goes back to [Jabin, Vega, *JMPA*, 2004]).

Idea of the proof

- Consider

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0.$$

- Kinetic form

$$\partial_t \chi(t,x,v) + v\partial_x \chi(t,x,v) \circ dw_t = \partial_v m(t,x,v).$$

- Rewrite as, for $\lambda > 0$,

$$\partial_t \chi(t,x,v) + v\partial_x \chi(t,x,v) \circ dw_t + \lambda\chi(t,x,v) = \partial_v m(t,x,v) + \lambda\chi(t,x,v).$$

Idea of the proof

- Change of variables

$$\chi(t,x,v) = e^{-\lambda t}\chi(0,x-vw_{0,t},v) + \int_0^t e^{-\lambda(t-s)}(\partial_v m)(s,x-vw_{s,t},v)\,ds + \lambda\int_0^t e^{-\lambda(t-s)}\chi(s,x-vw_{s,t},v)\,ds.$$

- Introduce the 'random' X-ray transform

$$T_t g(t,x) := \int_0^t g(s,x-vw_{s,t},v)e^{-\lambda(t-s)}\,ds$$

- Hence (neglecting the initial condition),

$$u := \int_v \chi = T(\partial_v m) + \lambda T\chi.$$

  where m is a finite measure and $\chi(t,x,v) := 1_{v<u} - 1_{v<0}$.
- Strategy: Estimate the regularity of $T(\partial_v m)$, $T\chi$ then use real interpolation.

Idea of the proof

Estimating the regularity of $T\chi$:

- Note $\partial_v \chi = \delta_{v=0} - \delta_{v=u(t,x)}$ and thus $\chi \in L^\infty_t L^1_{x,loc}(BV_v)$.
- Main point:

$$\partial_x g(s, x - vw_{s,t}, v) = -\frac{1}{w_{s,t}}(\partial_v(g(s, x - vw_{s,t}, v)) + \frac{1}{w_{s,t}}(\partial_v g)(s, x - vw_{s,t}, v).$$

- Velocity average

$$\int_v \partial_x g(s, x - vw_{s,t}, v) = \frac{1}{w_{s,t}} \int_v (\partial_v g)(s, x - vw_{s,t}, v).$$

- Hence,

$$\begin{aligned} \| \int_v g(s, x - vw_{s,t}, v)\|_{\dot{W}^{1,1}_{x,loc}} &= \|\partial_x \int_v g(s, x - vw_{s,t}, v)\|_{L^1_{x,loc}} \\ \leq \frac{1}{|w_{s,t}|} \| \int_v (\partial_v g)(s, x - vw_{s,t}, v)\|_{L^1_{x,loc}} &\leq \frac{1}{|w_{s,t}|} \|g\|_{L^\infty_t L^1_{x,loc} BV_v} \end{aligned}$$

Idea of the proof

- Thus,

$$\begin{aligned}\|Tg\|_{L^1_t \dot W^{1,1}_{x,loc}} &\le \int_0^T \left\| \int_0^t \int_v g(s,x-vw_{s,t},v)e^{-\lambda(t-s)}ds \right\|_{\dot W^{1,1}_{x,loc}} dt \\ &\le \int_0^T \int_0^t \| \int_v g(s,x-vw_{s,t},v)\|_{\dot W^{1,1}_{x,loc}} e^{-\lambda(t-s)}\, dsdt \\ &\le \|g\|_{L^\infty_t L^1_{x,loc} BV_v} \left(\int_0^T \int_0^t \frac{1}{|w_{s,t}|} e^{-\lambda(t-s)}\, dsdt \right).\end{aligned}$$

- Need to interpolate with trivial bound to compensate $\frac{1}{|w_{s,t}|}$ factor: For all $\sigma \in [0,1)$

$$\|Tg\|_{L^1_t \dot W^{\sigma,1}_{x,loc}} \le \|g\|_{L^\infty_t L^1_{x,loc} BV_v} \underbrace{\left(\int_0^T \int_0^t \frac{1}{|w_{s,t}|^\sigma} e^{-\lambda(t-s)}\, dsdt \right)}_{\text{suppose scaling in } \lambda}.$$

Path-by-path scaling condition

- *Path-by-path scaling condition*: Assume that there is a $\iota \in [\frac{1}{2}, 1]$ such that for every $\sigma \in [0,1)$, $\lambda \geq 1$ we have

$$\int_0^T \int_0^{T-r} e^{-\lambda t} |\underbrace{w_{t+r} - w_r}_{=:w_{r,r+t}}|^{-\sigma} \, dtdr \lesssim \lambda^{-1+\iota\sigma}.$$

- Easy to see: (ρ, γ)-irregularity implies path-by-path scaling.

Idea of the proof

- Using the scaling condition

$$\|Tg\|_{L^1_t \dot{W}^{\sigma,1}_{x,loc}} \leq \|g\|_{L^\infty_t L^1_{x,loc} BV_v} \left(\int_0^T \int_0^t \frac{1}{|w_{s,t}|^\sigma} e^{-\lambda(t-s)} dsdt \right)$$
$$\leq \|g\|_{L^\infty_t L^1_{x,loc} BV_v} \lambda^{-1+\iota\sigma}$$

Lemma

For $\sigma \in [0,1)$ we have

$$T : L^\infty_t(L^1_{x,loc}(BV_v)) \to L^1_t(\dot{W}^{\sigma,1}_{x,loc})$$

with

$$\|T\|_{L^\infty_t(L^1_{x,loc}(BV_v)) \to L^1_t(\dot{W}^{\sigma,1}_{x,loc})} \lesssim \lambda^{-1+\iota\sigma}.$$

Lemma

w is η-Hölder-continuous,

$$T(\partial_v m) = \partial_x h^1 + h^2$$

with

$$\|h^1\|_{L^1_t \mathscr{M}_x} \lesssim \lambda^{-1-\eta} \|m\|_{\mathscr{M}_{t,x}\mathscr{M}_{v,loc}}$$
$$\|h^2\|_{L^1_t \mathscr{M}_x} \lesssim \lambda^{-1} \|m\|_{\mathscr{M}_{t,x}\mathscr{M}_{v,loc}},$$

- From
$$u := \int_v \chi = T(\partial_v m) + \lambda T\chi.$$
- we get
$$u \in (L^1_t(\dot{W}^{-1,1}_x), L^1_t(\dot{W}^{\sigma,1}_{x,loc}))_{\theta,\infty} \subseteq L^1_t(\dot{W}^{s,1}_{x,loc})$$

  with $\theta = \frac{1+\eta}{\iota\sigma+1+\eta}$,
$$s < \sigma\left(\frac{1+\eta-\iota}{1+\eta+\iota}\right).$$

Theorem

Let u be a bounded quasi-solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{R}$$

and suppose that $w \in C^\eta([0,T])$ satisfies path-by-path scaling. Then, for all $\lambda < \frac{1+\eta-\iota}{1+\eta+\iota}$,

$$\|u\|_{L^1_t W^{\lambda,1}_x} < \infty.$$