Stochastic dynamics induced by porous media equations with linear multiplicative space-time noise

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1. From deterministic to stochastic dynamics
2. Porous medium equation perturbed by linear-multiplicative noise
3. Long-time behavior and random attractors

From deterministic to stochastic dynamics

Solutions deterministic differential equations

$$dX_t = f_0(X_t)dt$$

generate (semi-)flows, i.e. $\varphi : \mathbb{R}_+ \times H \to H$ with

$$\begin{aligned} \varphi(0) &= Id \\ \varphi(t+s) &= \varphi(t) \circ \varphi(s), \quad \forall s,t \geq 0. \end{aligned}$$

From deterministic to stochastic dynamics

Stochastic differential equations

$$\begin{aligned} dX_t(\omega) &= f_0(X_t(\omega))dt + \sum_{j=1}^{N} f_j(X_t)dW_t^j(\omega) \\ \text{``} &= f_0(X_t(\omega))dt + \sum_{j=1}^{N} f_j(X_t(\omega))\dot{W}_t^j(\omega)dt\text{``} \end{aligned} \tag{1.1}$$

with $X_0 = x$.

Obstacle: X_t solves (1.1) for each x, $\mathbb{P}$-a.s., no ω-wise solution. Existence of (semi-)flow $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \times H \times \Omega \to H$ unclear.

From deterministic to stochastic dynamics

Solution: $dim(H) < \infty$

- Stochastic equations can be 'lifted' to equations of homeomorphisms, i.e. solved for all x simultaneously.
- Kolmogorov continuity theorem to prove continuous dependence on x.

Result: Flow $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \times H \times \Omega \to H$ of homeomorphisms satisfying

$$\varphi(s, s; \omega) = Id$$
$$\varphi(t, s; \omega) = \varphi(t, r; \omega) \circ \varphi(r, s; \omega), \quad \forall s \leq r \leq t.$$

and $X_t(\omega) := \varphi(t, s; \omega)x$ solves the stochastic equation.

From deterministic to stochastic dynamics

Wiener noise as dynamical system: $\Omega = C(\mathbb{R};\mathbb{R}^N)$, $\mathbb{P}$ Wiener measure, $\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t)$.

Cocycle property: Flows to stochastic differential equations satisfy

$$\varphi(t,s;\omega)x = \varphi(t-s,0;\theta_s\omega)x$$

Random dynamical system: $\varphi : \mathbb{R}_+ \times H \times \Omega \to H$ measurable satisfying

- $\varphi(0;\omega) = Id$
- $\varphi(t+s;\omega) = \varphi(t;\theta_s\omega) \circ \varphi(s;\omega)$

e.g. Oseledets ergodic theorem.

From deterministic to stochastic dynamics

∞-**dimensional case**:

- Kolmogorov continuity theorem does not apply.
- Do not expect flow of homeomorphisms

PME in a random environment

Porous medium equation in a random environment

PME in a random environment

- Stochastic fluctuation of the porosity of the medium:

$$\partial_t \rho = \underbrace{c(\xi_t)\Delta(\rho^m)}_{=:A(\rho,\xi)}, \;\; m > 1.$$

- Expansion of first order in the noise:

$$\partial_t \rho = c\Delta\rho^m + D_\xi A(\rho, 0)\xi_t$$

- We get the stochastic porous medium equation with multiplicative noise

$$dX_t = c\Delta X_t^m dt + B(X_t)dW_t.$$

- Affine linear Taylor approximation of diffusion coefficients

$$dX_t = c\Delta X_t^m dt + B(0)dW_t + \underbrace{DB(0)(X_t)dW_t}_{=\sum_{k=1}^\infty f_k X_t d\beta_t^k}.$$

- Non-negativity

$$dX_t = c\Delta X_t^m dt + \sum_{k=1}^\infty f_k X_t d\beta_t^k.$$

Let $\mathcal{O}$ be a bounded domain. We consider

$$dX_t = \Delta X_t^m dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}, \text{ on } [0,T] \times \mathcal{O},$$

$$dX_t = \Delta X_t^m dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } [0,T] \times \mathcal{O},$$

with Dirichlet boundary conditions, $f_k \in C^\infty(\bar{\mathcal{O}})$.

Let $\mu_t(\xi) = -\sum_{k=1}^N f_k(\xi)\beta_t^{(k)} \mu_t(\xi) = -\sum_{k=1}^N f_k(\xi) z_t^{(k)}$. The transformation $Y = e^\mu X$ yields:

$$\frac{dY_t}{dt} = e^{\mu_t}\Delta(e^{-m\mu_t} Y_t^m). \qquad \text{(TPME)}$$

Partial construction: [BR11][1], for bounded initial data.

[1] V. Barbu, M. Röckner, *On a random scaled Porous Medium Equation, JDE, 2011*

Theorem

Let $Y_0 \in L^1(\mathcal{O})$. There is a unique map $Y \in C([0,T];L^1(\mathcal{O}))$ satisfying

- $Y \in L^\infty([\tau,T]\times\mathcal{O})$, $\forall \tau > 0$ *and*

$$\frac{dY}{dt} = e^{\mu_t}\Delta\left(e^{-m\mu_t}Y_t^m\right)$$

  for a.e. $t \in [0,T]$ *as an equation in* $H := (H_0^1(\mathcal{O}))^*$.

- L^1*-contractivity:*

$$\sup_{t\in[0,T]} \|Y_t^{(1)} - Y_t^{(2)}\|_{L^1(\mathcal{O})} \le C\|Y_0^{(1)} - Y_0^{(2)}\|_{L^1(\mathcal{O})}.$$

Theorem

Essentially bounded distributional solutions to (TPME) *are unique.*

Separate variable solutions

Deterministic PME:

$$\frac{dU}{dt} = \Delta U_t^m.$$

Let $U_t(\xi) = T(t)f(\xi)$. If $T' = T^m$ and $f \geq \Delta f^m$ then U is a supersolution. E.g.

$$U_t(\xi) = A\underbrace{t^{-\frac{1}{m-1}}}_{T(t)}\underbrace{(R^2 - |\xi|^2)^{\frac{1}{m}}}_{f(\xi)}.$$

Perturbed PME:

$$\frac{dU}{dt} = e^{\mu_t}\Delta\left(e^{-m\mu_t}U_t^m\right).$$

Let $U_t(\xi) = T(t)f(\xi)$ with $T' = T^m$ then U is a supersolution if

$$f \geq e^{\mu_t}\Delta\left(e^{-m\mu_t}f^m\right)$$

$$e^{-\mu_t}f \geq \Delta\left(e^{-\mu_t}f\right)^m$$

Substitute $g = fe^{-\mu_{\tau_i}}$:

$$e^{\mu_{\tau_i}-\mu_t}g \geq \Delta\left(e^{\mu_{\tau_i}-\mu_t}g\right)^m$$

Separate variable solutions

Theorem

There is a function $U : (0, T] \to \mathbb{R}_+$ *(*$U(0) \equiv \infty$*), piecewise smooth on* $(0, T]$ *such that*

$$|Y_t| \leq U_t, \text{ a.e. in } \mathcal{O},$$

for all $t \in [0, T]$*, independent of the initial condition* $Y_0 \in L^1(\mathcal{O})$*.*

Theorem

- $\varphi(t,\omega)x := X(t,0;\omega)x = e^{-\mu_t} Y(t,0;\omega)x$ *defines a continuous RDS on* $L^1(\mathcal{O})$.
- φ *is quasi-weakly-continuous on* $L^p(\mathcal{O})$, $p \in [1,\infty)$ *and quasi-weakly***-continuous on* $L^\infty(\mathcal{O})$.
- φ *is order preserving (i.e.* $\varphi(t,\omega)x_1 \le \varphi(t,\omega)x_2$, *a.e. in* $\mathcal{O}$ *if* $x_1, x_2 \in L^1(\mathcal{O})$ *with* $x_1 \le x_2$ *a.e. in* $\mathcal{O}$*).*

Random Attractors

Long-time behavior and random attractors

(Random) Attractors

Reduction of complexity: Try to find a subset $\mathcal{A} \subseteq H$ such that

- $\mathcal{A}$ is invariant.
- $\mathcal{A}$ captures the complete dynamics for large times, i.e. each trajectory can be approximated arbitrarily well by one lying in $\mathcal{A}$.
- $\mathcal{A}$ is minimal.

Then $\mathcal{A}$ is said to be a (random) attractor.

Random Attractors

Usual approach to prove existence of random attractors:

- Prove ω-wise attraction by a bounded set.
- Based on this prove ω-wise attraction by a compact set.

Recall:

$$\begin{aligned} dY_t &= e^{\mu_t}\Delta\left(e^{-m\mu_t}Y_t^m\right) \\ &= \Delta\left(e^{(1-m)\mu_t}Y_t\right) - 2\nabla e^{\mu_t}\cdot\nabla(e^{-m\mu_t}Y_t^m) - e^{-m\mu_t}Y_t^m\Delta e^{\mu_t}. \end{aligned}$$

Recall:

Theorem

There is a function $U : [0, T] \to \bar{\mathbb{R}}$ *(*$U(0) \equiv \infty$*), piecewise smooth on* $(0, T]$ *such that*

$$|Y_t| \leq U_t, \text{ a.e. in } \mathcal{O},$$

for all $t \in [0, T]$*, independent of the initial condition* $Y_0 \in L^1(\mathcal{O})$*.*

Asymptotic Compactness

We choose the approximations in a way that the results from [DB83][2] proving (local) uniform continuity of solutions are applicable.
A quantity is said to depend only on the data if it is a function of d, m, T, $\|X_0\|_{L^\infty(\mathcal{O})}$.

Theorem

Let $X_0 \in L^1(\mathcal{O})$ and X be the corresponding solution. Then

- *X is continuous on every compact set $K \subseteq (0,T] \times \mathcal{O}$, with modulus of continuity depending only on the data and $\mathrm{dist}(K, \partial\mathcal{O}_T)$.*
- *Assume:*

  ($\mathcal{O}1$) *There exist $\theta^* > 0, R_0 > 0$ such that $\forall x_0 \in \partial\mathcal{O}$ and $\forall R \leq R_0$:* $|\mathcal{O} \cap B_R(x_0)| < (1-\theta^*)|B_R(x_0)|$.

  Then for every $\tau > 0$, X is continuous on $[\tau, T] \times \bar{\mathcal{O}}$ with modulus of continuity depending only on the data, θ^* and τ.

[2] E. DiBenedetto, *Continuity of weak solutions to a general porous medium equation*, Indiana Univ. Math. J., 1983

In particular: The stochastic, variational solution X to

$$dX_t = \Delta X_t^m dt + \sum_{k=1}^{N} f_k X_k \circ d\beta_t^k$$

is $\mathbb{P}$-a.s. $L^\infty(\mathcal{O})$ bounded on each interval $[\tau, T]$, $\tau > 0$ and uniformly continuous on each compact set $K \subseteq (0, T] \times \mathcal{O}$.

Theorem

The RDS φ has a random attractor $\mathcal{A}$ as an RDS on $L^1(\mathcal{O})$. $\mathcal{A}$ is compact and attracting in each $L^p(\mathcal{O})$, $p \in [1, \infty)$.

$\mathcal{A}(\omega)$ is a bounded set in $L^\infty(\mathcal{O})$ and the functions in $\mathcal{A}(\omega)$ are equicontinuous on every compact set $K \subseteq \mathcal{O}$.

If $(\mathcal{O}1)$ is satisfied, then $\mathcal{A}(\omega)$ is compact and attracting in $L^\infty(\mathcal{O})$.

Outlook

- Finer structure of the attractor (dimension, invariant manifolds etc.)
- Finite speed of propagation and rates
- Expansion of the support and rates