Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise

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Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\mathcal{O}$ in arbitrary dimension $d \in \mathbb{N}$, $T > 0$ and $\mathcal{O}_T := [0, T] \times \mathcal{O}$. We consider

$$dX_t = \Delta(|X_t|^m sgn(X_t))dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T \qquad (0.1)$$
$$X(0) = X_0, \text{ on } \mathcal{O}$$

$m > 1$, with Dirichlet boundary conditions, driving signals $z^{(k)} \in C([0, T]; \mathbb{R})$, $f_k \in C^\infty(\bar{\mathcal{O}})$.

Generation of Stochastic Flows

Problem: When do SPDE generate stochastic flows? Show

$$X(t,s;\omega)x = X(t,r;\omega)X(r,s;\omega)x.$$

Technique: Transform the SPDE into an ω-wise random PDE.
Obstacle: Depending on the structure of the noise the random PDE becomes difficult so solve. Works for:

1. additive noise ($B(X_t) = \text{const}$),
2. real linear multiplicative noise ($B(X_t) = cX_t$),

We consider linear multiplicative space-time noise

$$B(X_t) = \sum_{k=1}^{N} f_k(\xi) X_t dz_t^{(k)}$$

for stochastic porous media equations

$$dX_t = \Delta(|X_t|^m sgn(X_t))dt + B(X_t) \circ dz_t. \qquad \text{(RPME)}$$

With $\mu_t(\xi) = -\sum_{k=1}^{N} f_k(\xi) z_t^{(k)}$ the transformation $Y = e^{\mu} X$ yields:

$$dY_t = e^{\mu_t} \Delta(e^{-m\mu_t} |Y_t|^m sgn(Y_t)). \qquad \text{(TPME)}$$

Partial construction: [BR10][1].

[1] V. Barbu, M. Röckner, *On a random scaled Porous Medium Equation, preprint, 2010*

Pathwise Solution

Theorem

Essentially bounded distributional solutions to (TPME) *are unique.*

Theorem

Let $Y_0 \in L^1(\mathcal{O})$, $z \in C([0,T];\mathbb{R}^N)$. There is a unique map $Y \in C([0,T];L^1(\mathcal{O}))$ satisfying

1. $Y \in L^\infty([\tau,T]\times\mathcal{O})$, $\Phi(e^{-\mu}Y) \in L^1([\tau,T];H_0^1(\mathcal{O}))$, $\forall \tau > 0$ *and*
$$\frac{dY}{dt} = e^{\mu_t}\Delta\Phi(e^{-\mu_t}Y_t)$$
for a.e. $t \in [0,T]$ *as an equation in* $H := (H_0^1(\mathcal{O}))^*$.
2. L^1*-contractivity:*
$$\sup_{t\in[0,T]} \|(Y_t^{(1)} - Y_t^{(2)})^+\|_{L^1(\mathcal{O})} + \|(\Phi(e^{-\mu}Y^{(1)}) - \Phi(e^{-\mu}Y^{(2)}))^+\|_{L^1(\mathcal{O}_T)}$$
$$\leq C\|(Y_0^{(1)} - Y_0^{(2)})^+\|_{L^1(\mathcal{O})}.$$

Generation of RDS

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, $(z_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^N$-valued adapted stochastic process and $((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. Assume

- $(S1)$ (Strictly stationary increments) For all $t, s \in \mathbb{R}$, $\omega \in \Omega$:
  $z_t(\omega) - z_s(\omega) = z_{t-s}(\theta_s\omega) - z_0(\theta_s\omega)$.
- $(S2)$ (Regularity) z_t has continuous paths.

We consider:

$$dX_t = \Delta\Phi(X_t)dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T \qquad \text{(SPME)}$$

$$X(0) = X_0, \text{ on } \mathcal{O}.$$

Theorem

$\varphi(t,\omega)x := X(t,0;\omega)x$ defines a continuous RDS and φ is order preserving (i.e. $\varphi(t,\omega)x_1 \le \varphi(t,\omega)x_2$, a.e. in $\mathcal{O}$ if $x_1, x_2 \in L^1(\mathcal{O})$ with $x_1 \le x_2$ a.e. in $\mathcal{O}$).

Random Attractors

Usual approach to prove existence of random attractors:

1. Prove ω-wise attraction by a bounded set via the transformed equation.
2. Based on this prove ω-wise attraction by a compact set via the transformed equation.

Recall

$$
\begin{aligned}
dY_t &= e^{\mu_t}\Delta(\Phi(e^{-\mu_t}Y_t)) \\
&= e^{(1-m)\mu_t}\Delta(\Phi(Y_t)) + 2e^{\mu_t}\nabla e^{-m\mu_t}\cdot\nabla\Phi(Y_t) + \Phi(Y_t)e^{\mu_t}\Delta e^{-m\mu_t}.
\end{aligned}
$$

Coercivity of porous medium operator is not preserved. Thus usual approach to prove bounded absorption via coercivity is not applicable.

Solution: Construct explicit supersolution with initial value ∞ and bounded for all $t > 0$. The construction combines an interval splitting technique from [BR10][2] and the known deterministic case:

$$K(t,\xi) = At^{-\frac{1}{m-1}}(R^2 - |\xi|^2)^{\frac{1}{m}}.$$

Theorem

There is a function $U : [0, T] \to \bar{\mathbb{R}}$ *(*$U(0) \equiv \infty$*), piecewisely smooth on* $(0, T]$ *such that*

$$|Y_t| \leq U_t, \text{ a.e. in } \mathcal{O},$$

for all $t \in [0, T]$*, independent of the initial condition* $Y_0 \in L^1(\mathcal{O})$*.*

[2] V. Barbu, M. Röckner, *On a random scaled Porous Medium Equation, preprint, 2010*

Asymptotic Compactness: We choose the approximations in a way that the results from [DB83][3] proving (local) uniform continuity of solutions are applicable.

[3] E. DiBenedetto, *Continuity of weak solutions to a general porous medium equation*, Indiana Univ. Math. J., 1983

We say that a quantity depends only on the data if it is a function of d, m, T, $\|X_0\|_{L^\infty(\mathcal{O})}$.

Theorem

Let $z \in C([0,T];\mathbb{R}^N)$, $X_0 \in L^1(\mathcal{O})$ and X be the corresponding solution. Then

1. *X is uniformly continuous on every compact set $K \subseteq \mathring{\mathcal{O}}_T$, with modulus of continuity depending only on the data and dist$(K, \partial\mathcal{O}_T)$.*
2. *Assume:*

   ($\mathcal{O}1$) *There exist $\theta^* > 0, R_0 > 0$ such that $\forall x_0 \in \partial\mathcal{O}$ and $\forall R \leq R_0$: $|\mathcal{O} \cap B_R(x_0)| < (1-\theta^*)|B_R(x_0)|$.*

   Then for every $\tau > 0$, X is uniformly continuous on $[\tau, T] \times \bar{\mathcal{O}}$ with modulus of continuity depending only on the data, θ^* and τ.

In particular: The stochastic, variational solution X to

$$dX_t = \Delta(|X_t|^m sgn(X_t))dt + \sum_{k=1}^{N} f_k X_k \circ d\beta_t^k$$

is $\mathbb{P}$-a.s. $L^\infty(\mathcal{O})$ bounded on each interval $[\tau, T]$, $\tau > 0$ and uniformly continuous on each compact set $K \subseteq (0, T] \times \mathcal{O}$.

Theorem

The RDS φ has a random attractor A.

$A(\omega)$ is a bounded set in $L^\infty(\mathcal{O})$ and the functions in $A(\omega)$ restricted to any compact set $K \subseteq \mathring{\mathcal{O}}$ are uniformly bounded and equicontinuous on K.

If $(\mathcal{O}1)$ is satisfied, then $A(\omega)$ is a set of uniformly bounded and equicontinuous functions on $\bar{\mathcal{O}}$.