# Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise [2].

**Benjamin Gess**

International Graduate College “Stochastics and Real World Models”, Beijing — Bielefeld

bgess@math.uni-bielefeld.de

## 1. Introduction

Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\mathcal{O}$ in arbitrary dimension $d \in \mathbb{N}$, $T > 0$ and $\mathcal{O}_T := [0,T] \times \mathcal{O}$. We consider partial differential equations driven by rough signals of the type

$$\begin{aligned} dX_t &= \Delta(|X_t|^m sgn(X_t))dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T \\ X(0) &= X_0, \text{ on } \mathcal{O} \end{aligned} \tag{1}$$

with Dirichlet boundary conditions, $m > 1$, driven by signals $z^{(k)} \in C([0,T];\mathbb{R})$ and (for simplicity) $f_k \in C^\infty(\bar{\mathcal{O}})$. Giving meaning to equation (1), in particular to the occurring stochastic integral is part of the results.

Due to the spatial dependency of the functions $f_k$ the noise acts in space as well as in time. For this type of noise even the generation of a continuous RDS has been an open problem and is solved here ([2]) for the first time. In contrast to the case of additive or real (i.e. non-spatially distributed) multiplicative noise, the standard method of transforming the SPDE into a random PDE becomes non-trivial, since the space-dependency of the noise destroys the monotonicity structure of the transformed equation.

## 2. Construction of (pathwise) solutions

The construction of solutions to (1) for signals of bounded variation proceeds by first transforming the equation into a PDE, then constructing solutions to this transformed equation. More precisely, let $\mu_t(\xi) := -\sum_{k=1}^N f_k(\xi) z_t^{(k)}$. Then $Y := e^{\mu} X$ satisfies the transformed equation

$$\partial_t Y_t = e^{\mu_t}\Delta((e^{-\mu_t}Y_t)^m sgn(e^{-\mu_t}Y_t)), \text{ on } \mathcal{O}_T \tag{2}$$

with Dirichlet boundary conditions and initial condition $Y_0$. This transformation will be rigorously justified below. Our results extend [1] where under restrictions on the dimension $d$ and the order $m$ unique existence of solutions for (2) with essentially bounded initial conditions has been shown.

Let us define what we mean by a solution to (1) and (2). Let $W^{n,p}(\mathcal{O})$ be the Sobolev space of order $n$ in $L^p(\mathcal{O})$, $W_0^{n,p}(\mathcal{O})$ the subspace of functions vanishing on $\partial\mathcal{O}$, $C^{m,n}(\bar{\mathcal{O}}_T) \subseteq C(\bar{\mathcal{O}}_T)$ be the set of all continuous functions on $\mathcal{O}_T$ having $m$ continuous derivatives in time and $n$ continuous derivatives in space and let $C^{1\text{-var}}([0,T];H)$ be the set of functions of bounded variation. We define $H_0^1(\mathcal{O}) := W_0^{1,2}(\mathcal{O})$ and denote its dual by $H$. Further, let $\Phi(r) := |r|^m sgn(r)$.

**Definition 1.** *For $Y_0 \in L^1(\mathcal{O})$ we call $Y \in L^1(\mathcal{O}_T)$ a (very) weak solution to (2) if $\Phi(e^{-\mu}Y) \in L^1([0,T];W_0^{1,1}(\mathcal{O}))$ ($\in L^1(\mathcal{O}_T)$ resp.) and*

$$-\int_{\mathcal{O}_T} Y_r\, \partial_r \eta \, d\xi dr - \int_{\mathcal{O}} Y_0 \eta_0 \, d\xi = \int_{\mathcal{O}_T} \Phi(e^{-\mu_r}Y_r)\Delta(e^{\mu_r}\eta_r)\, d\xi dr,$$

*for all $\eta \in C^1(\bar{\mathcal{O}}_T)$ ($\in C^{1,2}(\bar{\mathcal{O}}_T)$ resp.) with $\eta = 0$ on $[0,T] \times \partial\mathcal{O}$ and on $\{T\} \times \mathcal{O}$.*

For $z \in C^{1\text{-var}}([0,T];\mathbb{R}^N)$ (very) weak solutions to (1) are defined similarly. A rigorous formulation for the transformation of (1) into (2) can be given as following: Let $X_0 \in L^1(\mathcal{O})$, $z \in C^{1\text{-var}}([0,T];\mathbb{R}^N)$ and $X \in L^1(\mathcal{O}_T)$ with $X \in C([0,T];H)$ or $X \in C([0,T];L^1(\mathcal{O}))$. Then $X$ is a very weak solution to (1) iff $Y := e^{\mu}X$ is a very weak solution to (2). We prove

**Theorem 2.** *Essentially bounded very weak solutions to (2) are unique.*

**Theorem 3.** *Let $Y_0 \in L^\infty(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$. There exists a unique weak solution $Y \in C([0,T];H) \cap L^\infty(\mathcal{O}_T)$ to (2) satisfying $\Phi(e^{-\mu}Y) \in L^2([0,T];H_0^1(\mathcal{O}))$. There is a function $U : [0,T] \to \bar{\mathbb{R}}$ (taking the value $\infty$ at $t = 0$) which is piecewise smooth on $(0,T]$ such that for all $Y_0 \in L^\infty(\mathcal{O})$*

$$Y_t \le U_t, \text{ a.e. in } \mathcal{O}, \ \forall t \in [0,T].$$

If $z \in C^{1\text{-var}}([0,T];\mathbb{R}^N)$ then this yields the existence of a weak solution to (1) given by $X = e^{-\mu}Y$. A key point of Theorem 3 is that the upper bound $U_t$ does not depend on the initial condition $Y_0$.

Solutions to (1) for continuous signals are constructed by an approximation of the driving signal.

**Definition 4.** *Let $z \in C([0,T];\mathbb{R}^N)$. We call $X \in C([0,T];H)$ a rough weak solution to (1) if $X(0) = X_0$ and for all approximations $z^{(\varepsilon)} \in C^{1\text{-var}}([0,T];\mathbb{R}^N)$ of the driving signal $z$ with $z^{(\varepsilon)} \to z$ in $C([0,T];\mathbb{R}^N)$ and corresponding weak solutions $X^{(\varepsilon)}$ to (1) driven by $z^{(\varepsilon)}$ we have $X_t^{(\varepsilon)} \to X_t$ in $H$ for all $t \in [0,T]$.*

**Theorem 5.** *Let $X_0 \in L^\infty(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$. Then there exists a unique rough weak solution $X$ to (1) given by $X = e^{-\mu}Y$, where $Y$ is the corresponding weak solution to (2). $X$ satisfies $X_t \le U_t$ a.e. in $\mathcal{O}$ for all $t \in [0,T]$, where $U$ is as in Theorem 3.*

By proving Lipschitz continuity in the initial condition with respect to the $L^1(\mathcal{O})$ norm we obtain the existence of solutions to (1) for initial conditions in $L^1(\mathcal{O})$ in a limiting sense. Let $C^w([0,T];H)$ be the space of weakly continuous maps into $H$.

**Definition 6.** *Let $X_0 \in L^1(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$. A function $X \in C^w([0,T];L^1(\mathcal{O}))$ is said to be a limit solution to (1) if $X(0) = X_0$ and for all approximations $X_0^{(\delta)} \in L^\infty(\mathcal{O})$ with $X_0^{(\delta)} \to X_0$ in $L^1(\mathcal{O})$ and corresponding rough weak solutions $X^{(\delta)}$ to (1) we have $X_t^{(\delta)} \to X_t$ in $L^1(\mathcal{O})$ uniformly in time.*

**Theorem 7.** *Let $z \in C([0,T];\mathbb{R}^N)$. For each $X_0 \in L^1(\mathcal{O})$ there is a unique limit solution $X$ to (1) satisfying $\Phi(X) \in L^1(\mathcal{O}_T)$. For $X_0^{(i)} \in L^1(\mathcal{O})$, $i = 1,2$ the corresponding limit solutions satisfy*

$$\sup_{t\in[0,T]} \|(X_t^{(1)} - X_t^{(2)})^+\|_{L^1(\mathcal{O})} + \|(\Phi(X^{(1)}) - \Phi(X^{(2)}))^+\|_{L^1(\mathcal{O}_T)} \le C\|(X_0^{(1)} - X_0^{(2)})^+\|_{L^1(\mathcal{O})}.$$

*In addition, $X_t \le U_t$ a.e. in $\mathcal{O}$ for all $t \in [0,T]$, where $U_t$ is as in Theorem 3.*

As a special application we obtain a comparison principle: For $X_0^{(1)}, X_0^{(2)} \in L^1(\mathcal{O})$ with $X_0^{(1)} \le X_0^{(2)}$ a.e. we have $X_t^{(1)} \le X_t^{(2)}$, for all $t \in [0,T]$, a.e. in $\mathcal{O}$.

## 3. Regularity of solutions

We say that a quantity depends only on the data if it is a function of $d$, $m$, $T$. We assume

$$\begin{aligned} &\text{There exist } \theta^* > 0, R_0 > 0 \text{ such that } \forall x_0 \in \partial\mathcal{O} \text{ and } \forall R \le R_0: \\ &|\mathcal{O} \cap B_R(x_0)| < (1-\theta^*)|B_R(x_0)|. \end{aligned} \tag{$\mathcal{O}1$}$$

By proving that regularity results due to Emmanuele DiBenedetto may be applied in our situation we obtain

**Theorem 8.** *Let $z \in C([0,T];\mathbb{R}^N)$, $X_0 \in L^1(\mathcal{O})$ and $X$ be the corresponding limit solution. Then*

1. *For every $\tau > 0$, $X$ is uniformly continuous on $[\tau,T] \times \bar{\mathcal{O}}$ with modulus of continuity depending only on the data, $\theta^*$ and $\tau$.*
2. *If $X_0 \in L^\infty(\mathcal{O})$ is continuous on a compact set $K \subseteq \mathcal{O}$, then $X$ is uniformly continuous on $[0,T] \times K'$ for every compact set $K' \subseteq \mathring{K}$, with modulus of continuity depending only on the data, dist$(K,\partial\mathcal{O})$, dist$(K',\partial K)$, $\|X_0\|_{L^\infty(\mathcal{O})}$ and the modulus of continuity of $X_0$ over $K$.*

## 4. Generation of RDS

We now pass to the case of stochastically perturbed porous media equations. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, $(z_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^N$-valued adapted stochastic process and $((\Omega,\mathcal{F},\mathbb{P}),(\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. We assume

(Strictly stationary increments) $z_t(\omega) - z_s(\omega) = z_{t-s}(\theta_s\omega)$, $\forall t,s \in \mathbb{R}$, $\omega \in \Omega$.

(Regularity) $z_t$ has continuous paths.

We have assumed $z_0 = 0$ for notational convenience only. In particular, applications include fractional Brownian Motion with arbitrary Hurst parameter. We then consider the SPDE

$$\begin{aligned} dX_t &= \Delta\Phi(X_t)dt + \sum_{k=1}^N f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T \\ X(0) &= X_0, \text{ on } \mathcal{O}. \end{aligned} \tag{3}$$

For $x \in L^1(\mathcal{O})$ and $\omega \in \Omega$ let $X(t,s;\omega)x$ denote the solution to (1) with initial value $x$ at time $s$ driven by the continuous signal $z_\cdot(\omega)$.

If the signal $z$ is given by a continuous semimartingale then (3) can be interpreted in the sense of stochastic Stratonovich integration. In this case we show that the limit solution $X$ is a probabilistic solution to (3). Together with the pathwise convergence of the approximants $X^{(\varepsilon)} \to X$ obtained in Theorem 5 via approximation by paths of bounded variation this yields a pathwise Wong-Zakai result.

**Theorem 9.** *The map $\varphi$ given by $\varphi(t-s,\theta_s\omega)x := X(t,s;\omega)x$ ($t \ge s$, $\omega \in \Omega$, $x \in L^1(\mathcal{O})$) is a continuous RDS and $\varphi$ is order preserving, i.e. $\varphi(t,\omega)x_1 \le \varphi(t,\omega)x_2$ a.e. in $\mathcal{O}$ if $x_1, x_2 \in L^1(\mathcal{O})$ with $x_1 \le x_2$ a.e. in $\mathcal{O}$.*

## 5. Existence of a random attractor

Let $\mathcal{D}$ be the system of all random closed sets. The RDS $\varphi$ satisfies the same regularity and regularizing properties as in Theorem 8. Using this we prove

**Theorem 10.** *The RDS $\varphi$ has a $\mathcal{D}$-random attractor $A$ (as an RDS on $L^1(\mathcal{O})$). $A$ is compact in each $L^p(\mathcal{O})$ and attracting in $L^p(\mathcal{O})$-norm, $p \in [1,\infty)$. Moreover, $A(\omega)$ is a bounded set in $L^\infty(\mathcal{O})$ and the functions in $A(\omega)$ restricted to any compact set $K \subseteq \mathcal{O}$ are equicontinuous on $K$. If ($\mathcal{O}1$) is satisfied, then $A(\omega)$ is compact in $C(\bar{\mathcal{O}})$ and attracting in $L^\infty(\mathcal{O})$-norm.*

## References

[1] Viorel Barbu and Michael Röckner. On a random scaled porous medium equation. *to appear in: J. Differential Equations*, pages 1–22, 2011.

[2] Benjamin Gess. Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise. *arXiv:1108.2413v1*, 2011.